Milnor's Fibration Theorem for Real Singularities

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Introduction

In 1968 John Milnor published his now classic "Singular points of complex hypersurfaces" [Mil68]. In this he proved the following:

Theorem. *Let f be a holomorphic function from $\mathbb{C}^n$ to $\mathbb{C}$ with singular value at 0. Let $V = f^{-1}(0)$ Then, for small enough $\varepsilon > 0$, we have a fiber bundle from $\mathbb{S}_\varepsilon^{2n-1} \setminus V$ in $\mathbb{C}^n$ to the circle $\mathbb{S}^1$ in $\mathbb{C}$ given by $f(p)/\|f(p)\|$.*

This is called Milnor's fibration theorem, and is a very important result in singularity theory.

One question of interest, which Milnor wrote about in his book, was whether this result could be extended to real analytic functions as well. If f is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ with $n \geq m$, can we find properties for f such that we have a fiber bundle from $\mathbb{S}_\varepsilon^{n-1} \setminus V$ in $\mathbb{R}^n$ to the unit sphere $\mathbb{S}^{m-1}$ in $\mathbb{R}^m$?

If f has an isolated singular point at 0, Milnor proved that for small enough spheres $\mathbb{S}_\varepsilon^{n-1}$ in $\mathbb{R}^n$ and $\mathbb{S}_\delta^{m-1}$ in $\mathbb{R}^m$, one can find a fiber bundle from the "Milnor-tube" $f^{-1}(\mathbb{S}_\delta^{m-1}) \cap \overline{\mathbb{B}}_\varepsilon^n$ to the sphere $\mathbb{S}_\delta^{m-1}$ given by f. This tube can be "blown up" to the sphere to give a fibration from $\mathbb{S}_\varepsilon^{n-1} \setminus V$ to $\mathbb{S}_\delta^{m-1}$. However, this fiber bundle can not always be given by $f(p)/\|f(p)\|$. The search continued, both for properties less restrictive than an isolated singular point and for properties which would give $f(p)/\|f(p)\|$ as a fiber bundle.

In [Hir77], it was proved by Hironaka that all holomorphic functions from $\mathbb{C}^n$ to $\mathbb{C}$ satisfy a condition called the Thom a_f-condition, defined in 2.3.6 on page 19. Lê used this in [Lê77] to prove that when f is holomorphic from $\mathbb{C}^n$ to $\mathbb{C}$ one does not need an isolated singular point for f to be a fiber bundle from the Milnor-tube to the sphere. Therefore, this fiber bundle has often been called the Milnor-Lê fibration. In the complex case the Milnor fibration and the Milnor-Lê fibration are equivalent fibrations. But even if both of these fibers exist when f is a real analytic function, we can't always prove that they are equivalent. The following questions are therefore of interest. When do the Milnor-Lê fibration exist? When do the Milnor fibration exist? And when are these two equivalent?

Lê needed both the Thom a_f-condition and isolated singular value at 0 to prove the existence of the Milnor-Lê fibration in the complex case. These properties were therefore a natural fit when one wanted to prove the equivalent statement in the real case. If f is a function with an isolated singular point or if f

is a holomorphic function from $\mathbb{C}^n$ to $\mathbb{C}$, both of these properties are fulfilled, so this encompassed all proven situations.

What we will ask for is not the Thom a_f-condition, but rather a property which we will call the Transversality condition. The transversality condition, from definition 1.4.5 on page 7, is implied by the Thom a_f condition but is not equivalent to it. It was first used by Araújo dos Santos, Chen and Tibăr in [AdSCT13], but they noted that they had no examples of functions satisfying the transversality condition that did not satisfy the Thom a_f-condition. Examples, however, do exist. We will see that the function $f : \mathbb{R}^3 \to \mathbb{R}^2$ given by $f(x, y, z) = (x, y(x^2 + y^2) + xz^2)$ do satisfy the transversality condition (see Example 1.4.9 on page 10), but do not satisfy the Thom a_f-condition (see 2.3.9 on page 20).

For the Milnor fibration to exist, we will need both the transversality condition and a property called d-regularity (see Definition 1.4.5 on page 7). This was shown to be sufficient in [AdSCT13]. Araújo dos Santos, Chen and Tibăr used the Ehresmann Fibration theorem (see 1.2.3 on page 3) for manifolds with boundary to show that for small enough ε and δ, $f/\|f\|$ is a fiber bundle on $\mathbb{S}^{n-1}_\varepsilon \setminus f^{-1}(\mathbb{B}^m_\delta)$ and $f/\|f\|$ is a fiber bundle on $\mathbb{S}^{n-1}_\varepsilon \cap f^{-1}(\overline{\mathbb{B}}^m_\delta \setminus \{0\})$. We can then glue these fiber bundles together, as they agree on the intersection. We will show that $f/\|f\|$ is a fiber bundle more directly. Our proof is reminiscent of the proof for the Ehresmann Fibration Theorem found in [Dun13, Lemma 9.5.8]. This is done in Theorem 3.3.3 on page 28.

The main theorem in [CMSS10] is that if we have d-regularity, the Thom a_f-condition and isolated singular value, then the Milnor fibration and the Milnor-Lê fibration are equivalent. However, a crucial step needed to show this is not adequately proven and I have not been able to construct an adequate proof. In [CMSS10, Lemma 5.2] they need that two vectors w_f and $w_{\mathfrak{F}}$ never point in opposite directions, but this is not fully proven. We will instead study two vector fields $\pi(p)$ and $\pi(\Lambda(p))$. That these never point in opposite directions is both necessary and sufficient to prove [CMSS10, Lemma 5.2], as we see in 3.3.1 on page 26. We will give some properties on f which will make sure that these never point in opposite directions. These properties are studied in Lemma 3.4.1 on page 31, Lemma 3.4.2 on page 32, and Lemma 3.4.3 on page 33.

Throughout this thesis we will try to build up enough theory to study these properties and see why they are sufficient. This will be done through three chapters.

Chapter 1, **Manifold Theory** will be about the theory of manifolds. It is assumed that the reader is familiar with manifolds and basic concepts such as tangent bundles, differentials and gradients. It will consist of five sections.

The first, *Functions and Manifolds* will define analytic functions, analytic manifolds and describe the most basic manifolds we will work with, the sphere and the ball. Here we will also meet our example function $f(x, y, z) = (x, y(x^2 + y^2) + xz^2)$ for the first time. This function will follow us through the thesis.

The second section, *Fiber Bundles* will define the concept of a fiber bundle.

As Milnor's fibration theorem is about the existence of fiber bundles, this is an important section. We will also see an example of a fiber bundle, the Jet bundle, which will be of use later. Lastly we will look at Ehresmann fibration theorem. This theorem shows that if f is a proper surjective submersion, then f is a fiber bundle.

Section three, *Singularities* will be a short section about singular points and singular values. We will mention a geometric version of the implicit function theorem, define isolated singular points, isolated singular values, and compute the singularities of the example function f we saw in section one.

In section four, *Transversality*, we will define transversal manifolds and functions. We then define a set which we will call the Milnor set of a function with the help of the concept of transversality. This set will be used to define the properties of d-regularity and the transversality condition, which we will need to show the existence of the Milnor fibration and the Milnor-Lê fibration. Then we will look at some alternate ways to compute whether these conditions are fulfilled. We end by showing that both our example function f and general holomorphic functions from $\mathbb{C}^n$ to $\mathbb{C}$ satisfy these conditions.

The last section, *Fold points*, will be about a type of singular points called fold points. These are defined with the help of both Jet bundles and the concept of transversality of functions. We will later, in Lemma 3.4.3 on page 33, show that if both the Milnor fibration and the Milnor-Lê fibration exists for function f, and the function $F(p) = (f(p), \|p\|)$ has only fold points as singularities, then the two fibrations will be equivalent.

In Chapter 2, **Analytic and Semianalytic Sets**, we will, as the headline implies, define both analytic sets and semianalytic sets. These are sets described locally by the zero sets of analytic functions. As we need to study the set $V = f^{-1}(0)$, some theory on analytic sets will be very helpful. The chapter consists of three sections. Most of the information contained in this chapter is from [ML07].

In the first section, *Analytic sets*, we will define what an analytic set is. We will define regular, singular and exceptional points in analytic sets and give some theorems about the properties of analytic sets.

The second section, *Semianalytic sets*, will give properties of semianalytic sets. These are defined by both zero sets of analytic functions and inequalities in analytic functions. The most important part of this section is the Curve Selection Lemma. The lemma shows that when we have a semianalytic set with 0 in its closure, we can always find an analytic curve $\gamma(t)$, starting at 0, contained in the set for all small values of $t \neq 0$. We end the section by using this to prove that for analytic functions with one-dimensional target, all singular values are isolated.

In section three, *Stratification*, we will study how we can use manifold theory on analytic and semianalytic sets by partitioning them into manifolds and study how these fit together. For us, the primary use of this is to define the concept of the Thom a_f-condition. The Thom a_f condition will be interesting to us as it is a property that all holomorphic functions from $\mathbb{C}^n$ to $\mathbb{C}$ have, and when combined with isolated singular value, it is sufficient to prove the existence of the Milnor-Lê fibration for real analytic functions. We will instead use the transversality condition, but the Thom a_f-condition implies the transversality condition, so any function which satisfies the Thom a_f-condition will be sufficient. We end this section by proving that our example function $f(x,y,z) = (x, y(x^2+y^2)+xz^2)$ do not satisfy the Thom a_f-condition. Therefore, f is an example of a function satisfying the transversality condition, but not the Thom a_f-condition.

The last chapter of the thesis, **Milnor Fibrations**, will be about the existence and equivalence of both the Milnor-Lê fibration and the Milnor fibration. We will give sufficient conditions for these to exist, show under which circumstances we can prove that they are equivalent and give some examples. This chapter consists of four sections. Much of the information in this chapter is from [Sea07].

In the first section, *The Milnor-Lê fibration*, we will state and prove a theorem which shows when we can expect existence of the Milnor-Lê fibration from the Milnor-tube to the sphere, both for functions from $\mathbb{C}^n$ to $\mathbb{C}$ with $f(0) = 0$ and for functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ with $f(0) = 0$. It will turn out that we need both isolated singular value at 0 and the transversality condition. These are both automatically fulfilled for any function from $\mathbb{C}^n$ to $\mathbb{C}$ with singular value at 0. We quickly discuss other properties that have been used as sufficient for the theorem.

Section two is about *Inflation to the sphere*. We can "blow up" the Milnor-Lê fibration from the Milnor-tube to the sphere along vector fields. This gives us a fibration from a sphere in $\mathbb{R}^n$ (or $\mathbb{C}^n$) to the sphere in $\mathbb{R}^m$ (or $\mathbb{C}$). The way this is done is by finding a smooth vector field v flowing outward from both the Milnor-tube and the sphere. We follow the flow of this vector field to get a fiber bundle on $\mathbb{S}_\varepsilon^{n-1} \setminus f^{-1}(\mathbb{B}_\delta^m)$, then glue this together with the Milnor-Lê fibration defined on $\mathbb{S}_\varepsilon^{n-1} \cap f^{-1}(\overline{\mathbb{B}}_\delta^m \setminus \{0\})$ to get a fibration defined everywhere on $\mathbb{S}_\varepsilon^{n-1} \setminus f^{-1}(0)$.

In the third section, *The Milnor fibration*, we will prove the existence of the Milnor fibration $f/\|f\|$ from $\mathbb{C}^n$ to $\mathbb{C}$. This will be proven by inflating the Milnor-Lê fibration along vector fields preserving the argument of $f(z)$. This will also show that the Milnor-Lê fibration and the Milnor fibration are equivalent, and it will give us a platform to work from when we look for the properties we need to show equivalence in the real case. We will then show that if we assume the transversality condition and d-regularity, we have the Milnor fibration for real maps $f : \mathbb{R}^n \to \mathbb{R}^m$. Note, however, that we will not assume isolated singular value. Therefore, there may exist functions f which has the Milnor fibration, but not the Milnor-Lê fibration. This result is the main theorem in [AdSCT13], and is the first instance I have seen where we have used the transversality condition instead of using the Thom a_f-condition.

Lastly, in *Equivalence of fibrations*, we will go through under what circumstances we can expect these fibrations to be equivalent. In [CMSS10] they try to prove that the fibrations will be equivalent if we have the Thom a_f-condition, d-regularity and isolated singular value. Again, the Thom a_f-condition is only needed to get the transversality condition. We already needed the transversality condition, d-regularity and isolated singular value to insure the existence of both fibrations, so this would imply that if both fibrations existed, they would always be equivalent. However, as mentioned earlier, a crucial lemma, [CMSS10, Lemma 5.2], is not proven adequately, and I have not been able to fill the gap. Under the assumption that two specific vectors, $\pi(p)$ and $\pi(\Lambda(p))$ never point in opposite directions, we can prove the lemma. Therefore, we take this assumption as an extra condition. We then show under which circumstances we can expect this condition to be fulfilled. Among other things, we see that if the function $F(p) = (f(p), \|p\|)$ has only fold points as singular points, we get equivalence of fibrations. Then we will show that for our example function $f(x, y, z) = (x, y(x^2 + y^2) + xz^2)$, $\pi(p)$ and $\pi(\Lambda(p))$ will indeed not point in different directions, and we'll end by going through some extra examples.

Chapter 1

Manifold theory

1.1 Functions and manifolds

We are going to study the fibers of functions between manifolds. But if we let our functions take any possible form, this quickly becomes impossible to work with, so we want our functions to have some kind of structure. Therefore we will often assume that our functions are analytic or holomorphic. In this section, we will let $\mathbb{K}$ be shorthand for either $\mathbb{R}$ or $\mathbb{C}$, as many of our definitions will work just as well with both fields.

Definition 1.1.1. Let U and V be open sets, $U \subset \mathbb{K}^n$ and $V \subset \mathbb{K}^m$. Let f be a function from U to $\mathbb{K}$. Let ϕ be a function from V to $\mathbb{K}^n$. Then

(i) f is analytic, $f \in C^\omega_\mathbb{K}(U)$, if the Taylor series of f about each point in U converges to f in a neighborhood of that point,

(ii) and ϕ is analytic, $\phi \in C^\omega_\mathbb{K}(V, \mathbb{K}^n)$, if $f \circ \phi$ is analytic for all $f \in C^\omega_\mathbb{K}(U)$, for all U such that $\phi(V) \subset U$.

If a function is complex analytic, we will call it holomorphic.

Often we are only interested in how a function behaves close to a point, so rather than studying a function, we might want to study its function germ.

Definition 1.1.2. Let U and V be open in $\mathbb{K}^n$. Let $f \in C^\omega_\mathbb{K}(U, \mathbb{K}^m)$ and $g \in C^\omega_\mathbb{K}(V, \mathbb{K}^m)$. We define an equivalence relation $\sim_x$ at a point $x \in U \cap V$ by $f \sim_x g$ if there exists an open neighborhood $W \subset U \cap V$ of x such that $f(y) = g(y)$ for all $y \in W$. Then $C^\omega_{\mathbb{K},x}(\mathbb{K}^n, \mathbb{K}^m)$ is the collection of equivalence classes for all U and V, called the **analytic stalk of** x. A **germ at** x is an element in the stalk.

Intuitively the germ describes the behavior of a function f on arbitrarily small neighborhoods of x. The concept of a germ could be defined for continuous or smooth functions as well, but the analytic stalk of a point is especially easy to describe, as it is all power series that converge in a neighborhood around x.

For analytic function germs, we have an important result called the principle of analytic continuation.

Theorem 1.1.3. *Let p be a point in $\mathbb{K}^n$ and let U be an open, connected neighborhood of p. Assume $f \in C^\omega_K(U, \mathbb{K}^m)$. If the equivalence class of f in $C^\omega_{\mathbb{K},x}(\mathbb{K}^n, \mathbb{K}^m)$ is 0, then f is the constant function equal to 0.*

So if f and g are equal on an open neighborhood of a point, then the germ of $f - g$ is 0, so we must have that f and g are equal everywhere both are defined.

As our first example of an analytic function, we will look at a function which is actually polynomial in nature.

Example 1.1.4. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(x, y, z) = (x, y(x^2 + y^2) + xz^2)$. This is an analytic function, as all polynomials are. We will study this function in greater detail later, as it fulfills many of the properties we will define.

As we want to study functions between manifolds, analytic manifold will occur naturally.

Definition 1.1.5. Let M be a topological manifold with an atlas $U = \{(U_i, x_i)|i \in I\}$, where $x_i : U_i \to \mathbb{K}^{n_i}$. Let $U_{ij} = U_i \cap U_j$ and let $x_{ij} = x_i \circ x_j^{-1} : x_j(U_{ij}) \to x_i(U_{ij})$. Then M is an analytic manifold if $x_{ij} \in C^\omega_{\mathbb{K}}(x_i(U_{ij}), \mathbb{K}^{n_j})$ for all $i, j \in I$.

We will, as in the case with functions, call a complex analytic manifold for a holomorphic manifold. In the case where $n_i = 2$ for all i, it is often called a **Riemann surface**.

Two manifolds in particular will be of interest to us. The first is the n-sphere of radius δ, $\mathbb{S}^n_\delta$, given by $\mathbb{S}^n_\delta = \{x \in \mathbb{R}^{n+1}|\|x\| = \delta\}$. The second is the open n-ball of radius δ, $\mathbb{B}^n_\delta$, given by $\mathbb{B}^n_\delta = \{x \in \mathbb{R}^n|\|x\| < \delta\}$. These are sometimes contained in $\mathbb{C}^n$ as well, where we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. If we do not write the δ, then δ is assumed to be 1.

1.2 Fiber Bundles

We will start this section with the definition of a fiber bundle.

Definition 1.2.1. Let T, F and M be smooth manifolds and let $\pi : T \to M$ be a submersion. Let $T_U = \pi^{-1}(U)$ for any subset U of M. Then π is a fiber bundle of M with fiber F if for every $p \in M$, there exists an open U_p and a diffeomorphism $\phi_{U_p} : T_{U_p} \to U_p \times F$ such that $\pi_{U_p} \circ \phi_{U_p} = \pi$, where $\pi_{U_p} : U_p \times F \to U_p$ is the projection in the first coordinate.

As an obvious example of fiber bundles, we see that the tangent bundle of a connected manifold is a fiber bundle where the fiber F is a vector space of dimension equal to the dimension of the manifold. Another trivial example is given by $T = F \times M$, called the trivial fiber bundle.

As another example of fiber bundles, let us define Jet bundles. The following definition is from [GG73].

Definition 1.2.2. Let M and N be smooth, connected manifolds with p a point in M. Let f and g be smooth functions from M to N with $f(p) = g(p) = q$.

(i) We say that f has first order contact with g at p if $df(p) = dg(p)$.

(ii) We say that f has kth order contact (for $k > 0$) with g at p if df has kth order contact with dg for all points in T_pM. We write $f \sim_k g$ at p.

(iii) Let $J^k(M,N)_{p,q}$ be the set of equivalence classes under $\sim_k$ at p of all functions $f : M \to N$ with $f(p) = q$.

(iv) Let $J^k(M,N) = \bigcup_{(p,q)\in M\times N} J^k(M,N)_{p,q}$. This is called the kth **Jet bundle**. An element σ of $J^k(M,N)$ is called a **k-jet** from M to N.

For a function $f : M \to N$ we have a natural map $j^k f : M \to J^k(M,N)$ which sends a point p in M to the σ in $J^k(M,N)_{p,q}$ represented by f.

If we let σ be a 1-jet in $J^1(M,N)_{p,q}$ represented by a function f, we can see that we get a canonical isomorphism between $J^1(M,N)$ and $\mathrm{Hom}(TM,TN)$ by sending σ to $df(p)$. It is then easy to see that this is a fiber bundle over $M \times N$ with fiber $\mathrm{Hom}(T_pM,T_qN)$ as all $\mathrm{Hom}(T_pM,T_qN)$ are diffeomorphic for all points $(p,q) \in M \times N$. This because $\mathrm{Hom}(T_pM,T_qN)$ is isomorphic to the set of matrixes of dimension $\dim N \times \dim M$.

Indeed, as the equivalence classes only depend on how the function f behaves locally, we can choose an open neighborhood of p diffeomorphic to $\mathbb{R}^{\dim M}$ and an open neighborhood of q diffeomorphic to $\mathbb{R}^{\dim N}$. Then the jet bundle $J^k(M,N)_{p,q}$ must be diffeomorphic to $J^k(\mathbb{R}^{\dim M},\mathbb{R}^{\dim N})_{0,0}$, so the jet bundles $J^k(M,N)_{p,q}$ will be diffeomorphic for all $(p,q) \in M \times N$. We then see that $J^k(M,N)$ is a fiber bundle over $M \times N$ with fibers diffeomorphic to $J^k(M,N)_{p,q}$.

There's a very important theorem regarding fiber bundles called the Ehresmann fibration theorem. This tells us when we can expect a given map to give us a fibration. It was originally proved for manifolds without boundary, but have been extended to manifolds with boundary as well. We will state the version for manifolds with boundary. This is from [Lê12].

Theorem 1.2.3. *Let $(M,\partial M)$ be a smooth manifold M with boundary ∂M. Let f be a smooth map onto a connected manifold N. If*

(i) the map f is proper,

(ii) the restriction of f to ∂M is submersive and surjective onto N, and

(iii) the restriction of f to M is submersive and surjective onto N,

then f is a fiber bundle.

1.3 Singularities

As we are interested in the fibers on manifolds, we want to know when we can assume that these fibers may indeed be interesting. We are studying the local structure of the fibers, so if the fiber itself is a manifold, then locally they just look like $\mathbb{R}^n$ for some n. Fibers that are not manifolds are therefore more interesting than fibers that are manifolds. What we end up looking for are singular values.

Definition 1.3.1. Let $f : M \to N$ be a function between manifolds M and N. If p is a point in M where f is not a submersion, i.e. df is not surjective, then p is a **singular point**. Otherwise p is a **regular point**. If q is a point in N such that there exists a singular point p in M with $f(p) = q$, then q is a **singular value**. Otherwise q is a **regular value**.

We will write Σf for the set of singular points of f.

The reason why singular values are more interesting to us than regular values comes from the following theorem, a geometric version of the implicit function theorem, from [ML07, Thm. 2.1].

Theorem 1.3.2. *Let $f : M \to N$ be a function between manifolds M and N with $\dim M = n$ and $\dim N = m$. Let $p \in M$ and let $q = f(p)$. If p is a regular point of f, then we may find open neighborhoods U of p and V of q with $f(U) \subset V$ and local coordinates $\phi : U \to R^n$ and $\psi : V \to R^m$, such that $\phi(p) = 0$, $\psi(q) = 0$ and $\psi \circ f \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}^m$ are given by projection down to the first m coordinates.*

Specifically, if q is a regular value, then $f^{-1}(q)$ will be a manifold of dimension $n - m$.

So if q is a regular value of f, then $f^{-1}(q)$ may be interesting globally, but locally all we see is $\mathbb{R}^{n-m}$. But on the other hand regular values are easier to work with. We can get an interesting local structure on $f^{-1}(q)$ and still use manifold theory on all fibers close to $f^{-1}(q)$ if the singular value q is isolated.

Definition 1.3.3. Let $f : M \to N$ be a function between manifolds M and N, and let p be a singular point of f on the manifold M. Then p is an **isolated singular point** if there exists an open neighborhood U of p where f is a submersion for all points $p' \in U \setminus \{p\}$.

Let q be a singular value of f in the manifold N. Then q is an **isolated singular value** if there exists an open neighborhood V of q where q' is a regular value for all $'q \in V \setminus \{q\}$.

We will now return to the function we studied earlier in example 1.1.4 on page 2.

Example 1.3.4. Let $f(x, y, z) = (x, y(x^2 + y^2) + xz^2)$. If we compute $df = \begin{bmatrix} \nabla x \\ \nabla(y(x^2 + y^2) + xz^2) \end{bmatrix}$ we get

$$df = \begin{bmatrix} 1 & 0 & 0 \\ 2xy + z^2 & x^2 + 3y^2 & 2xz \end{bmatrix}.$$

It's easy to see that this matrix will only have rank less than 2 if both $x^2+3y^2 = 0$ and $2xz = 0$. The first equation implies both $x = 0$ and $y = 0$ If $x = 0$, then the second equation is already fulfilled, so the set where f is not a submersion, Σf, must be the set where $x = 0$ and $y = 0$, i.e. the z-axis.

We see that if $f(x, y, z) = (0, 0)$, then we have $x = 0$ and $y(x^2 + y^2) + xz^2 = y^3 = 0$. If we let $V = f^{-1}(0)$, we see that V must then be the set of points where both $x = 0$ and $y = 0$. Then we have $\Sigma f = V$, so 0 is the only singular value. This is then an isolated singular value.

1.4 Transversality

The concept of transversality will be an important one for us. As we later want to examine when a function f is a fiber bundle, we often need more from our function than just analyticity. That f is transversal to spheres will turn out to be of great importance. The following definition is taken from [GG73].

Definition 1.4.1. Let M and N be manifolds, and let W be a submanifold of N. Let $f : M \to N$ be a smooth map and let p be a point in M. Then f and W intersect transversally at p, $f \pitchfork_p W$, if either

(a) $f(p) \notin W$ or

(b) $f(p) \in W$ and $T_{f(p)}N = T_{f(p)}W + (df)_p(T_pM)$

If A is a subset of M then f and W intersect transversally on A, $f \pitchfork_A W$, if $f \pitchfork_p W$ for all $p \in A$. Finally, f and W intersect transversally, $f \pitchfork W$, if $f \pitchfork_M W$.

We also have the concept of submanifolds intersecting transversally. If V and W are submanifolds of M and $i : V \to M$ is the inclusion, we say that V and W intersect transversally at p, $V \pitchfork_p W$ if $i \pitchfork_p W$. We then define $V \pitchfork_A W$ and $V \pitchfork W = V \pitchfork_M W$ as above.

If $f : M \to N$, $g : M \to K$ are functions from a manifold M to manifolds N and K, we say that f and g meets transversally at p, $f \pitchfork_p g$, if both f and g are submersions at p and $f^{-1}(f(p)) \pitchfork g^{-1}(g(p))$. This is well-defined thanks to Theorem 1.3.2 on the preceding page.

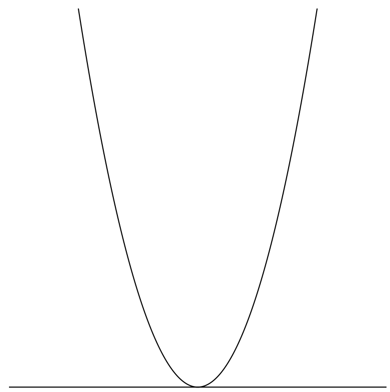

Figure 1.1: Non-transversal

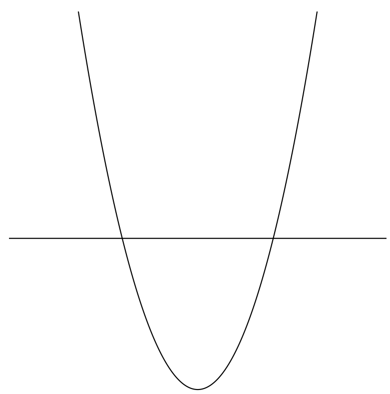

Figure 1.2: Transversal

To get a feeling of what this means geometrically, let us look at some simple examples.

Example 1.4.2. Let $f(t) = (t, t^2)$ be a function from $\mathbb{R}$ to $\mathbb{R}^2$. Let W be the submanifold of $\mathbb{R}^2$ given by $W = \{(x, y) \in \mathbb{R}^2 | y = 0\}$, and let W' be the submanifold given by $W' = \{(x, y) \in \mathbb{R}^2 | y = 4\}$.

For $t \in \mathbb{R}$, we have that $f(t) \in W$ only if $t = 0$. We have

$$df_t = \begin{bmatrix} 1 \\ 2t \end{bmatrix}$$

and therefore

$$df_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We see that $df_0(T_0\mathbb{R}) = T_{f(0)}W$, so $T_{f(0)}W + df_0(T_0\mathbb{R}) \neq T_{f(0)}\mathbb{R}^2$. Our function f and the manifold W are therefore not transversal, $f \not\pitchfork W$. See figure 1.1.

On the other hand, if $f(p) \in W'$, we have that $p_1 = -2$ or $p_2 = 2$. We then have

$$df_{-2} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

and

$$df_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

Then

$$T_{f(-2)}W' + df_{-2}(T_{-2}\mathbb{R}) = \text{span}\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \end{bmatrix}\} = T_{f(-2)}\mathbb{R}^2$$

and

$$ThisT_{f(2)}W' + df_{-2}(T_{-2}\mathbb{R}) = \text{span}\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \end{bmatrix}\} = T_{f(2)}\mathbb{R}^2.$$

The function f and manifold W' are therefore transversal, $f \pitchfork W'$. See figure 1.2 on the preceding page.

Functions transversal to manifolds behave nicely, as the following lemma, [GG73, Thm. 4.4], shows us.

Lemma 1.4.3. *Let M and N be smooth manifolds, W a submanifold of N. Let $f : M \to N$ be smooth and assume $f \pitchfork W$. Then $f^{-1}(W)$ is a submanifold of M with* $\operatorname{codim} f^{-1}(W) = \operatorname{codim} W$.

Of particular interest to us is the **Milnor Set** of a real analytic function, which are described by transversality.

Definition 1.4.4. Let $\mathbb{B}^n_\delta \subset \mathbb{R}^n$ be the open ball with center 0 and radius δ. Let $\rho : \mathbb{B}^n_\delta \to \mathbb{R}_{\geq 0}$ be the square of the euclidean distance from 0. The **Milnor Set** of an analytic function $f : \mathbb{B}^n_\delta \to \mathbb{R}^m$ is

$$M(f) = \{x \in \mathbb{B}^n_\delta : f \not\pitchfork_x \rho\},$$

the set of points in $\mathbb{B}^n_\delta$ where f and ρ do not meet transversally. Similarly, if $V = f^{-1}(0)$, the Milnor set of $f/\|f\| : \mathbb{B}^n_\delta \setminus V \to \mathbb{S}^{m-1}$ is

$$M(f/\|f\|) = \overline{\{x \in \mathbb{B}^n_\delta \setminus V : f/\|f\| \not\pitchfork_x \rho\}}$$

We are interested in when a function satisfies two specific conditions described by the Milnor sets, which we will call the **Transversality condition** and ***d*-regularity**.

Definition 1.4.5. Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $n \geq m$, be an analytic function germ of 0, with $f(0) = 0$. We will write $f(p) = (P_1(p), \ldots, P_m(p))$. Then f satisfies the **Transversality condition** if there exists a δ such that, when we take the Milnor set over $\mathbb{B}^n_\delta$,

$$\overline{M(f) \setminus V} \cap V \subseteq \{0\}.$$

We say that f satisfies d**-regularity** if there exists a δ such that

$$M(f/\|f\|) = \emptyset.$$

We want to examine how we can describe these sets and conditions more explicitly.

Proposition 1.4.6. *Let p be a point in $\mathbb{B}^n_\delta$ Then p is not contained in $M(f)$ if and only if the matrix*

$$A = \begin{bmatrix} df(p) \\ p \end{bmatrix} = \begin{bmatrix} \nabla P_1 \\ \vdots \\ \nabla P_m \\ p \end{bmatrix}$$

has rank $m+1$

Proof. We will let $T_p f = T_p f^{-1}(f(p))$. If $p = 0$, then ρ is not a submersion. Therefore p must be contained in $M(f)$, and we see that A will have rank less than $m+1$. So we can from now on assume that $p \neq 0$.

If p is not contained in $M(f)$, then f must be a submersion. Then df will have rank m and all we need to see is that p must be linearly independent of the gradients $\nabla P_1, \ldots, \nabla P_m$. But these gradients span the normal vectors to the tangent space $T_p f$ of $f^{-1}(f(p))$, and if p is in the span of these, then p is normal to $T_p f$. Therefore p must be normal to all of $\mathbb{R}^n$, as $T_p f \oplus T_p \mathbb{S}^{n-1}_{\|p\|} = \mathbb{R}^n$. But then we would have $p = 0$, which we have assumed not to be the case. Therefore, A must have rank $m+1$.

If A has rank $m+1$, we have that f is a submersion. We also have that ρ is a submersion, as ρ is a submersion for all $p \neq 0$. If $T_p f$ and $T_p \mathbb{S}^{n-1}_{\|p\|}$ were to be nontransversal, then as $T_p \mathbb{S}_{\|p\|}$ is $n-1$-dimensional, we must have $T_p f \subset T_p \mathbb{S}^{n-1}_{\|p\|}$. But then p must be normal to $T_p f$, and therefore in $\operatorname{span}\{\nabla P_1, \ldots \nabla P_m\}$. But, as A has rank $m+1$, this is not the case, so f and ρ must be transversal. $\square$

For the function $f/\|f\|$ we will first create a chart on the sphere $\mathbb{S}^{m-1}$. Let $r = (r_1, \ldots, r_m)$ be a point in $\mathbb{S}^{m-1}$ and assume for simplicity that $|r_m| \geq |r_i|$ for all $i = 1, \ldots m-1$. We may then define a chart by letting $(r_1, \ldots r_m) \mapsto (\frac{r_1}{r_m}, \ldots \frac{r_{m-1}}{r_m})$. If we restrict to the half-sphere containing r and bounded by the hyperplane $r_m = 0$, this is a chart, which we will call y.

We may use this chart to calculate $d(y \circ f/\|f\|)$. We then have

$$\begin{aligned} y \circ f/\|f\| &= \begin{bmatrix} \frac{P_1/\|f\|}{P_m/\|f\|} \\ \vdots \\ \frac{P_{m-1}/\|f\|}{P_m/\|f\|} \end{bmatrix} \\ &= \begin{bmatrix} P_1/P_m \\ \vdots \\ P_{m-1}/P_m \end{bmatrix} \end{aligned}$$

and therefore we get

$$d(y \circ f/\|f\|) = P_m^{-2} \begin{bmatrix} P_m \nabla P_1 - P_1 \nabla P_m \\ \vdots \\ P_m \nabla P_{m-1} - P_{m-1} \nabla P_m \end{bmatrix}$$

The set $M(f/\|f\|)$ can now be described by checking for what points the matrix

$$\begin{bmatrix} \Omega_1 \\ \vdots \\ \Omega_{m-1} \\ p \end{bmatrix}$$

has rank m, where $\Omega_i = P_m \nabla P_i - P_i \nabla P_m$, as in Proposition 1.4.6 on the facing page. We get the following lemma.

Lemma 1.4.7. *If f is d-regular, then there exists a small $\delta > 0$ such that the matrix*

$$A = \begin{bmatrix} \Omega_1 \\ \vdots \\ \Omega_{m-1} \\ p \end{bmatrix}$$

has rank m for all p in $\mathbb{B}_\delta^n \setminus V$ with $P_m(p) \neq 0$.

If there is a small delta > 0 such that the matrix A has rank Thism for all p in $\mathbb{B}_\delta^n \setminus V$ with $P_m(p) \neq 0$, for any permutation on the indexes $1, \ldots, m$, then f is d-regular.

With this description of d-regularity, we can get yet another alternate description of the Milnor-set of f. Let $\Lambda(p) = \sum_{j=1}^m P_j(p) \nabla P_j(p) = \frac{1}{2} \nabla \|f(p)\|^2$

Lemma 1.4.8. *If f is d-regular, a point p with $P_m(p) \neq 0$ is not in $M(f)$ if and only if*

$$A = \begin{bmatrix} \Omega_1 \\ \vdots \\ \Omega_{m-1} \\ \Lambda \\ p \end{bmatrix}$$

has rank $m + 1$.

Proof. We want to show that A has rank $m + 1$ if and only if the matrix

$$B = \begin{bmatrix} \nabla P_1 \\ \vdots \\ \nabla P_m \\ p \end{bmatrix}$$

has rank $m+1$.

We see that we have $\text{span}\{\Omega_1, \ldots, \Omega_{m-1}, \Lambda\} \subset \text{span}\{\nabla P_1, \ldots \nabla P_m\}$. Therefore, the rank of A must be less than or equal to the rank of B. So if B does not have rank $m+1$, then A cannot have rank $m+1$ either. We need to prove that if A has rank less than $m+1$, then B must have rank less than $m+1$ as well.

Assume that A has rank less than $m+1$, but B has rank $m+1$. As f is d-regular, we have that the matrix

$$\begin{bmatrix} \Omega_1(p) \\ \vdots \\ \Omega_{m-1}(p) \\ p \end{bmatrix}$$

has rank m. We can therefore write

$$\begin{aligned}
\sum_{j=1}^{m} P_j(p)\nabla P_j(p) &= \sum_{i=1}^{m-1}(a_i\Omega_i(p)) + bp \\
&= \sum_{i=1}^{m-1}(a_iP_m(p)\nabla P_i(p) - a_iP_i(p)\nabla P_m(p)) + bp \\
&= \sum_{i=1}^{m-1}(a_iP_m(p)\nabla P_i(p)) - \sum_{i=1}^{m-1}(a_iP_i(p))\nabla P_m(p) + bp
\end{aligned}$$

As we have assumed that B has rank $m+1$, we must have

$$\begin{aligned}
P_j(p) &= a_jP_m(p) \qquad \text{for j = 1, \ldots, m-1} \\
P_m(p) &= -\sum_{i=1}^{m-1} a_iP_i(p) \\
b &= 0
\end{aligned}$$

which gives us that $a_j = P_j(p)/P_m(p)$ and therefore $P_m(p)^2 = -\sum_{i=1}^{m-1} P_i(p)^2$. But as we have assumed that $P_m(p) \neq 0$, the left hand side here is positive while the right hand side is non-positive. This is impossible, so if A has rank less than $m+1$, then B must also have rank less than $m+1$. $\square$

Here we will return to our earlier example 1.3.4 on page 5, to see if it is d-regular or satisfies the transversality condition.

Example 1.4.9. The function $f : \mathbb{R}^3 \to \mathbb{R}^2$ is given by $f(x,y,z) = (x, y(x^2 + y^2) + xz^2)$. If $f(x,y,z) = (0,0)$, we see that we must have $x = 0$ and $y = 0$. The set $V = f^{-1}(0)$ is therefore the z-axis. The differential of f is

$$df = \begin{bmatrix} 1 & 0 & 0 \\ 2xy + z^2 & x^2 + 3y^2 & 2xz \end{bmatrix}.$$

Let us compute the set $M(f)$. If $p = (x, y, z)$ is in $M(f)$, then we have that the rank of the matrix

$$A = \begin{bmatrix} df \\ p \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2xy + z^2 & x^2 + 3y^2 & 2xz \\ x & y & z \end{bmatrix}$$

must be less than 3, i.e. the determinant of this matrix must be 0. We have that

$$\det A = (x^2 + 3y^2 - 2xy)z$$

so if $\det A = 0$ then either $z = 0$ or $x^2 + 3y^2 - 2xy = (x - y)^2 + 2y^2 = 0$.

If $(x - y)^2 + 2y^2 = 0$, then we must have $y = 0$ and $x - y = 0 \Rightarrow x = 0$.

Therefore we have

$$M(f) = \{(x, y, z) \in \mathbb{R}^3 | (x = 0 \wedge y = 0) \vee z = 0\},$$

i.e. $M(f)$ is the xy-plane and the z-axis. We can now easily see that $\overline{M(f) \setminus V} \cap V = 0$.

Let us now see if f is d-regular. We compute the matrix

$$B = \begin{bmatrix} \Omega_1(p) \\ p \end{bmatrix} = \begin{bmatrix} y^3 - x^2y & -x^3 - 3xy^2 & -2x^2y \\ x & y & z \end{bmatrix}.$$

This matrix has rank 2 if we can find a 2×2 submatrix with determinant different from 0. Let us compute the determinant of the first two columns, which we will call B_1. We get

$$\det B_1 = y^4 - x^2y^2 + x^4 + 3x^2y^2 = x^4 + y^4 + 2x^2y^2.$$

We see that if (x, y, z) is a point not in V, then either $x \neq 0$ or $y \neq 0$, so $\det B_1 > 0$. Therefore, we must have that f is d-regular as well.

Note that in this case we did not need to bother with the assumption that $P_m(p) = 0$, as a permutation of the indices would only give a $\Omega_1(p)$ differing by a constant (the constant -1). The linear (in)dependence of the row vectors $\Omega_1(p)$ and p would therefore have been unchanged.

The conclusion is that our function f is both d-regular and satisfies the transversality condition.

We will later be interested in holomorphic functions from $\mathbb{C}^n$ to $\mathbb{C}$, as these functions are the ones discussed in Milnor's original fibration theorem. It will therefore be interesting to note that these functions will fulfill both d-regularity and the transversality condition.

Lemma 1.4.10. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function germ with $f(0) = 0$. Then f is both d-regular and satisfies the transversality condition.*

The transversality condition will be shown later, see Proposition 2.3.8 on page 20.

To prove d-regularity, we need a lemma from Milnor, [Mil68, Lemma 4.3]

Lemma 1.4.11. *For every $z \in \mathbb{C}^n \setminus f^{-1}(0)$ sufficiently close to the origin, the two vectors z and $\nabla \log f(z)$ are either linearly independent over $\mathbb{C}$ or $\nabla \log f(z) = \lambda z$ where λ is a non-zero complex number with $|\arg \lambda| < \frac{\pi}{4}$.*

Proof of Lemma 1.4.10 on the previous page. If we look at $f(z) = P(z) + iQ(z)$ as a function from $\mathbb{R}^{2n}$ to $\mathbb{R}^2$, we want to show that z and $P\nabla Q - Q\nabla P$ are linearly independent.

The definition of the complex gradient of a function f is

$$\nabla f(z) = (\overline{\frac{\partial f}{\partial z_1}}, \ldots, \overline{\frac{\partial f}{\partial z_n}}).$$

We will use this to compute $i\nabla \log f(z) = i\frac{\nabla f(z)}{\overline{f(z)}}$.

For $j = 1, \ldots, n$, we have

$$\begin{aligned}
i\overline{\frac{\partial f}{\partial z_j}} &= i\overline{\frac{\partial(P+iQ)}{\partial(x_j+iy_j)}} \\
&= \frac{i}{2}\overline{(\frac{\partial P}{\partial x_j} + \frac{\partial Q}{\partial y_j} + i(\frac{\partial Q}{\partial x_j} - \frac{\partial P}{\partial y_j}))} \\
&= \frac{1}{2}(\frac{\partial Q}{\partial x_j} - \frac{\partial P}{\partial y_j} + i(\frac{\partial P}{\partial x_j} + \frac{\partial Q}{\partial y_j})) \\
\frac{i\overline{\partial f/\partial z_j}}{\overline{f(z)}} &= \frac{(\frac{\partial Q}{\partial x_j} - \frac{\partial P}{\partial y_j} + i(\frac{\partial P}{\partial x_j} + \frac{\partial Q}{\partial y_j}))(P+iQ)}{2(P-iQ)(P+iQ)} \\
&= \frac{P(\frac{\partial Q}{\partial x_j} - \frac{\partial P}{\partial y_j}) + iP(\frac{\partial P}{\partial x_j} + \frac{\partial Q}{\partial y_j}) - (Q(\frac{\partial P}{\partial x_j} + \frac{\partial Q}{\partial y_j}) + iQ(\frac{\partial P}{\partial y_j} - \frac{\partial Q}{\partial x_j}))}{2(P^2+Q^2)} \\
&= \frac{P(\frac{\partial Q}{\partial x_j} + i\frac{\partial Q}{\partial y_j}) - Q(\frac{\partial P}{\partial x_j} + i\frac{\partial P}{\partial y_j})}{P^2+Q^2}
\end{aligned}$$

where we used the Cauchy-Riemann equations to get the last equality. We now see that

$$i\nabla \log f(z) = \frac{P(z)\nabla Q(z) - Q(z)\nabla P(z)}{P(z)^2 + Q(z)^2},$$

so by Lemma 1.4.7 on page 9 d-regularity is then equivalent to $i\nabla \log f(z)$ and z being linearly independent over $\mathbb{R}$. But as Milnor's lemma tells us that these are indeed linearly independent over $\mathbb{R}$, we have proved what we wanted. □

For later use, we compute $\nabla \log f(z)$ as well. We get

$$\nabla \log f(z) = \frac{P\nabla P + Q\nabla Q}{P^2 + Q^2}.$$

1.5 Fold points

A special type of singular points, which will be of interest to us later, are called fold points. Recall the definition of jet bundles from Definition 1.2.2 on page 3.

Let M and N be manifolds, and assume $\dim M \geq \dim N$. For the jet bundle $J^1(M, N)$, the jets can be represented as matrices. We can therefore give each jet a rank equal to the rank of the corresponding matrix, which we know is well defined from general manifold theory. We can then study the subset S_1 of $J^1(M, N)$ consisting of all jets σ with rank $\dim N - 1$. The following theorem is from [GG73, Ch. II, Thm. 5.4].

Theorem 1.5.1. *S_1 is a submanifold of $J^1(M, N)$ with codimension $\dim M - \dim N + 1$.*

If f is a smooth function between M and N we can define $S_1 f = (j^1 f)^{-1}(S_1)$, where $(j^1 f)(p)$ is the equivalence class of $df(p)$. If we have that $j^1 f \pitchfork S_1$, then this is a submanifold of M with $\dim S_1 f = \dim N - 1$, thanks to lemma 1.4.3 on page 7.

We can now define fold points as follows.

Definition 1.5.2. Assume $\dim M \geq \dim N$, and let $f : M \to N$ be a smooth function satisfying $j^1 f \pitchfork S_1$. Then $p \in S_1 f$ is a **fold point** if $T_p S_1 f + \ker df_p = T_p M$.

We will call f a **submersion with folds** if the only singular points of f are fold points. We will then call $S_1 f$ for the **fold locus** of f.

Our first observation is the following simple lemma.

Lemma 1.5.3. *If f is a submersion with folds, then f is an immersion on the fold locus.*

Proof. This is easy to see as f restricted to $S_1 f$ goes from a manifold of dimension $\dim N - 1$ to a manifold of dimension $\dim N$. We have that $d(f|_{S_1 f})$ has rank $\dim N - 1$, so this is an immersion. □

Now, let f be a function germ from $\mathbb{R}^n$ to $\mathbb{R}^m$ with 0 as an isolated singular value, and let us study the function germ $F = (f(p), \|p\|^2)$. Let $V = f^{-1}(0)$. If there exists a small enough ε so that F is a submersion with folds on $\mathbb{B}^n_\varepsilon \setminus V$, then f has an interesting property.

Lemma 1.5.4. *If $F = (f, \|p\|^2)$ is a submersion with folds, then f is a local diffeomorphism on $M(f) \setminus V$.*

Proof. As $S_1 F$ is the set of points where

$$\begin{bmatrix} df(p) \\ 2p \end{bmatrix}$$

has rank $m+1-1=m$, and there are no points in $\mathbb{B}^n_\varepsilon \setminus V$ where the rank is lower than m, we see that $S_1F \setminus V = M(f) \setminus V$. As F is an immersion on S_1F, we have that $\dim \ker dF = n-m$. But if $v \in \ker dF$, then we have that $df(p)v = 0$, so $v \in \ker df$. We therefore have that $\ker dF \subset \ker df$. But f is a submersion everywhere on $M(f) \setminus V$, so $\dim \ker df = n-m$, and we have $\ker dF = \ker df$. Therefore f is an immersion on $M(f) \setminus V$ as well, and then f must be a local diffeomorphism. $\square$

Chapter 2

Analytic and Semianalytic Sets

2.1 Analytic Sets

This section will use the definitions given by [ML07], but will focus on analytic sets in $\mathbb{R}^n$ or $\mathbb{C}^n$. We will again let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$, where the fields are interchangeable.

In earlier sections we have at times needed to study the zero set of analytic functions. These sets are often of interest, so they have been given the name analytic sets. In this section we will give some basic information about these sets, before we move on to semianalytic sets. Recall the definition of $C^{\omega}_{\mathbb{K}}(\mathbb{K}^n)$ from Definition 1.1.1 on page 1.

Definition 2.1.1. Let $A \subset C^{\omega}_{\mathbb{K}}(\mathbb{K}^n)$. Then the **zero locus** of A, $V(A)$, is the set of points $V(A) \coloneqq \{x \in \mathbb{K}^n | f(x) = 0 \quad \forall f \in A\}$.

If $A = \{f_1, \ldots, f_j\}$ we will write $V(f_1, \ldots, f_j)$ instead of $V(A)$.

Definition 2.1.2. An **analytic set** X is a closed set in $\mathbb{K}^n$ such that for all $x \in X$, there exists an open neighborhood W of x in $\mathbb{K}^n$ and a finite collection $f_1, \ldots, f_j \in C^{\omega}_{\mathbb{K}}(W)$ such that $V(f_1, \ldots, f_j) = W \cap X$.

Definition 2.1.3. Let X be an analytic set. A point $p \in X$ is **smooth** of dimension d if there exists an open neighborhood W of p such that $W \cap X$ is an analytic submanifold of W of dimension d. A smooth point of X of the highest dimension is a **regular** point, a non-smooth point is a **singular** point and a non-regular point is an **exceptional** point.

We denote the set of all smooth points of X by $\mathring{X}$, the set of smooth points of dimension d by $\mathring{X}^{(d)}$, and the set of all singular points, the **singular locus**, by ΣX.

The following theorem is from [ML07, Thm. 5.14].

Theorem 2.1.4. *Let X be an analytic set. Suppose $0 \leq d \leq n$. X then has the following properties.*

(i) $\mathring{X}^{(d)}$ *is a d-dimensional analytic submanifold of* $\mathbb{K}^n$ *and is an open subset of* X*.*

(ii) The $\mathring{X}^{(d)}$ *are disjoint for different values of d,* $\mathring{X} = \mathring{X}^{(0)} \cup \ldots \cup \mathring{X}^{(n)}$*,* $\mathring{X}$ *is an analytic submanifold of* $\mathbb{K}^n$*, and* $\mathring{X}$ *is open in* X*.*

(iii) $\mathring{X}$ *is dense in* X*.*

(iv) ΣX *is a closed, nowhere dense subset of* X*.*

We see that if an analytic set contains only smooth points, it is a manifold. As an example, if $X = V(f)$, this happens if f is a submersion on all points in X.

2.2 Semianalytic sets

Analytic sets are more general than manifolds, but are still pretty well-behaved. Therefore we can still get some interesting information from them. But often we need to intersect our analytic sets with sets which may not be analytic, such as closed balls. This gives us semianalytic sets, which are more general than analytic sets, but still behaves well enough that we can work on them.

Definition 2.2.1. A subset X of $\mathbb{R}^n$ is a **semianalytic set** if, for all $p \in \mathbb{R}^n$, there exists an open neighborhood W of p such that $W \cap X$ is a finite union of subsets of the form

$$V(f_1, \ldots, f_k) \cap \{g_i(x) > 0, i = 1, \ldots, l\}$$

where $f_1, \ldots, f_k, g_1, \ldots g_l$ are analytic in W.

A subset X of $\mathbb{C}^n$ is a **semianalytic set** if, for all $p \in \mathbb{C}^n$, there exists an open neighborhood W of p such that $W \cap X$ is a finite union of subsets of the form

$$V(f_1, \ldots, f_k) \cap \{g_i(x) \neq 0, i = 1, \ldots, l\}$$

where $f_1, \ldots, f_k, g_1, \ldots, g_l$ are holomorphic in W.

Semianalytic sets have a number of nice properties, some of which are summarized in the following Theorem. The proof can be found in [BM88, Chapter 2]

Theorem 2.2.2. *Let* X *be a semianalytic subset of* $\mathbb{K}^n$*. Then*

(i) The collection of semianalytic subsets of $\mathbb{K}^n$ *is closed under finite unions, finite intersections and taking complements.*

(ii) Every connected component of X *is semianalytic.*

(iii) The family of connected components of X is locally finite.

(iv) X is locally connected.

(v) The closure and interior of X are semianalytic.

The properties of the connected components of a semianalytic set will be of particular use to us.

There's a very important lemma regarding semianalytic sets, the Curve Selection Lemma, which we will be in need of later. It exists in both a real and complex form, but we will be primarily interested in the real form.

Theorem 2.2.3 (Curve Selection Lemma)**.** *Let $X \subset \mathbb{R}^n \setminus \{0\}$ be a semianalytic set with $0 \in \overline{X}$. Then there exists an analytic curve $\gamma : [0, \varepsilon) \to \mathbb{R}^n$ with $\gamma(0) = 0$ and $\gamma((0, \varepsilon)) \subset X$.*

Let $X \subset \mathbb{C}^n \setminus \{0\}$ be a semianalytic set with $0 \in \overline{X}$. Then there exists a complex analytic curve $\gamma : \mathbb{B}^2_\varepsilon \to \mathbb{C}^n$ with $\gamma(0) = 0$ and $\gamma(\mathbb{B}^2_\varepsilon \setminus \{0\}) \subset X$.

Here we use $\mathbb{B}^2_\varepsilon$ as a subset of $\mathbb{C}$. A proof of this is found in [Łoj65].

We can use the curve selection lemma to prove that analytic functions with a one-dimensional target have isolated singular values.

Theorem 2.2.4. *Let U be a subset of $\mathbb{K}^m$ with $0 \in U$, and let $f : U \to \mathbb{K}$ be an analytic function. Then $f(0)$ is an isolated singular value.*

Proof. Assume that $f(p)$ is not an isolated singular value. Then we must have $p \in \overline{\Sigma f \setminus f^{-1}(f(p))}$. By the Curve Selection Lemma, we may find either a real or complex curve γ with $\gamma(0) = 0$ and $\gamma(t) \in \Sigma f \setminus f^{-1}(f(p))$ for all small $t \neq 0$. Then we must have that $(f \circ \gamma)'(t)$ is an analytic function which is zero for all small $t \neq 0$ and by continuity zero for $t = 0$. Therefore $(f \circ \gamma)(t)$ is constant, and as $f(\gamma(0)) = f(p)$, we must have that $f(\gamma(t)) = f(p)$ for all small t, but this is a contradiction. $\square$

The proof is easily extended to points $p \neq 0$ by instead studying $g(q) = f(p - q)$.

2.3 Stratification

When we study analytic and semi-analytic sets, we work with subsets of manifold, but where the subsets are not manifolds themselves. We can describe these, by partitioning them into pieces, where each piece is a manifold. We can then describe how these pieces fit together. By doing this, we may use manifold theory on these partitions.

Definition 2.3.1. Let X be a subset of $\mathbb{R}^n$. A collection $\mathcal{S} = \{S_\alpha\}_{\alpha \in I}$ of non-empty subsets of $\mathbb{R}^n$ is a **semianalytic partition of** X if

(i) X is the disjoint union $\cup_{\alpha \in I} S_\alpha$;

(ii) Each S_α is an analytic submanifold of $\mathbb{R}^n$ and a connected semianalytic subset of $\mathbb{R}^n$;

(iii) $\mathcal{S}$ is locally finite.

We call the partition a **stratification** if it satisfies the following condition: if $S_\alpha, S_\beta \in \mathcal{S}$ with $S_\alpha \neq S_\beta$ and $S_\alpha \cap \overline{S_\beta} \neq \emptyset$, then $S_\alpha \subset \overline{S_\beta}$ and $\dim S_\alpha < \dim S_\beta$. Each S_α is then called a **stratum**.

So when can we find such stratifications? Luckily, we have the following theorem from [Łoj65].

Theorem 2.3.2. *Let $\{X_i\}_{i \in I}$ be a locally finite collection of semianalytic subsets of $\mathbb{R}^n$. Then there exists a semianalytic stratification $\mathcal{S}$ of $\mathbb{R}^n$ which is compatible with $\{X_i\}_{i \in I}$, i.e. every X_i is a union of elements in $\mathcal{S}$.*

By letting $\{X_i\}_{i \in I}$ consist of two elements, namely a semianalytic subset X and $\mathbb{R}^n \setminus X$, we can always find a stratification of X by using the $S_\alpha \in \mathcal{S}$ with $S_\alpha \subset X$.

As we want to study how these strata fit together, we want to know how a stratum behaves as it approaches another stratum. We usually require the Whitney conditions to help us with this task.

Definition 2.3.3. Let $\mathcal{S}$ be a semianalytic partition of X. Let $S_\alpha, S_\beta \in \mathcal{S}$ and let $p \in S_\alpha$.

(i) A **Whitney a) sequence in $\mathbb{S}_\beta$ at** p is a sequence of points $p_i \in S_\beta$ such that $\lim_{i \to \infty} p_i = p$ and the tangent spaces $T_{p_i} S_\beta$ converge to some vector space T.

(ii) A **Whitney b) pair of sequences in** (S_β, S_α) **at** p is a pair of sequences $p_i \in S_\beta$ and $q_i \in S_\alpha$ such that $\{p_i\}$ is a Whitney a) sequence in S_β at p, $\lim_{i \to \infty} q_i = p$ and the lines defined by p_i and q_i, $\overline{p_i q_i}$, converge to some line ℓ.

(iii) The pair (S_β, S_α) satisfies the **Whitney a) condition at** p if, for all Whitney a) sequences $\{p_i\}$ in S_β at p, $T_p S_\alpha \subseteq \lim_{i \to \infty} T_{p_i} S_\beta$.

(iv) The pair (S_β, S_α) satisfies the **Whitney b) condition at** p if, for all Whitney b) pairs of sequences $\{p_i\}$ and $\{q_i\}$ in (S_β, S_α) at p, $\lim_{i \to \infty} \overline{p_i q_i} \subseteq \lim_{i \to \infty} T_{p_i} S_\beta$

If the Whitney a) condition is satisfied for all pairs (S_β, S_α), for all points $p \in S_\alpha$, we call the partition a **Whitney a) partition of X**. If the partition is a stratification, we will call it a **Whitney a) stratification**.

If the Whitney b) condition is satisfied for all pairs (S_β, S_α), for all points $p \in S_\alpha$, we call the partition a **Whitney b) partition of X**. But, in this case we have from [Mat12, Prop. 2.4, Corr. 10.5] that if a partition is a Whitney b) partition, it is also both a Whitney a) partition and a stratification, so we will of course call it a **Whitney stratification**.

We have the following strengthened version of Theorem 2.3.2 on the facing page, thanks again to [Łoj65].

Theorem 2.3.4. *Let $\{X_i\}_{i\in I}$ be a locally finite collection of semianalytic subsets of $\mathbb{R}^n$. Then there exists a semianalytic Whitney stratification $\mathcal{S}$ of $\mathbb{R}^n$ which is compatible with $\{X_i\}_{i\in I}$, i.e. every X_i is a union of elements in $\mathcal{S}$.*

The following part of [ML07, Thm. 7.14] will be of particular interest to us.

Theorem 2.3.5. *Let X be a semianalytic subset of $\mathbb{R}^n$ with Whitney stratification $\mathcal{S}$. Let $p \in X$. Then, for all sufficiently small $\varepsilon > 0$, $\mathbb{S}^{n-1}_\varepsilon(p)$ intersects all stratas of $\mathcal{S}$ transversely.*

If we let $f : \mathbb{R}^n \to \mathbb{R}^m$, $n \geq m$ be a real analytic function with $f(0) = 0$, we can give $V(f)$ a stratification $\mathcal{S} = \{S_\alpha\}$ such that in a neighborhood U of the origin, $(U \setminus V) \cup \bigcup_\alpha (S_\alpha \cap U)$ is a Whitney stratification.

The stratification of $V(f)$ will give us the following definition.

Definition 2.3.6. Let f be an analytic function with isolated singular value. Assume we have a Whitney stratification as above. Let $\{x_k\}$ be a sequence of points with $x_k \in U \setminus V$ for all k, converging to a point x in some S_α, and assume that $T_{x_k} f^{-1}(f(x_k))$ converges to a linear space T. If, for all such sequences x_k we have that $T_x S_\alpha \subset T$, we say that f satisfies the Thom a_f-condition.

We are unfortunately not as lucky with the existence of analytic functions satisfying the Thom a_f-condition as we have been with the existence of stratifications. It was proved in [Hir77] that if f is a holomorphic function from $\mathbb{C}^n$ to $\mathbb{C}$, it satisfies the Thom a_f-condition. For real analytic functions it will not in general be satisfied, but if a function has an isolated singular point, then the function satisfies the Thom a_f-condition.

Lemma 2.3.7. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $n \geq m$ be a real analytic function with an isolated singular point at 0. Then f satsifies the Thom a_f-condition.*

Proof. Choose any stratification of $V(f)$ with one of the strata as $S_0 = \{0\}$. If a sequence of points x_k converge to 0 with $T_{x_k} f^{-1}(f(x_k))$ converging to T, then we will have $T_0 S_0 \in T$, as $T_0 S_0$ will just be a point. Assume therefore that p is a point in $V(f)$ with $p \neq 0$.

By the implicit function theorem (Theorem 1.3.2 on page 4), we can find local coordinates ϕ and ψ such that $\hat{f} = \psi \circ f \circ \phi^{-1}$ is just the projection given by $\hat{f}(x_1, \ldots x_n) = (x_1, \ldots, x_{n-m}, 0, \ldots, 0)$. For any point q in a neighborhood of p, let $\hat{q} = \phi(q)$. By the definition of ψ, we have $\hat{p} = 0$. Let p_k be a sequence of points converging to p. We write $\hat{p}_k$ as $\hat{p}_k = (\hat{p}_{k,1}, \ldots, \hat{p}_{k,n})$. Then, for k large enough we have $T_{\hat{p}_k}\hat{f}^{-1}(\hat{f}(\hat{p}_k))$ given by $(\hat{p}_{k,1}, \ldots, \hat{p}_{k,n-m}) \times \mathbb{R}^m \subset \mathbb{R}^n$. As p_k tends to p, $T_{\hat{p}_k}\hat{f}^{-1}(\hat{f}(\hat{p}_k))$ will tend to $(0, \ldots, 0) \times \mathbb{R}^m = T_{\hat{p}}\phi(V(f))$. But as ϕ is a diffeomorphism, then we must have $T_pV(f) = T$, where T is the limit of $T_{p_n}f^{-1}(f(p_n))$. □

The reason we are interested in the Thom a_f-condition is because of [CMSS12, Prop. 3.3].

Proposition 2.3.8. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a real analytic function with isolated singular value satisfying the Thom a_f-condition at* 0*. Then there exists a small enough $\varepsilon > 0$ and an open set U of $V(f)$ such that for all $x \in U \setminus V(f)$ with $\|x\| = \varepsilon$, the manifold $f^{-1}(f(x))$ will intersect $\mathbb{S}^{n-1}_\varepsilon$ transversally.*

Proof. By theorem 2.3.5 on the preceding page, for small enough ε, the stratas S_α of $V(f)$ will intersect the sphere $\mathbb{S}^{n-1}_\varepsilon$ transversally. By the Thom a_f-condition, we can therefore find an open neighborhood U of $V(f)$ where $T_xf^{-1}(f(x)) + T_x\mathbb{S}^{n-1}_\varepsilon = T_x\mathbb{R}^n$ for all $x \in (U \setminus V(f)) \cap \mathbb{S}^{n-1}_\varepsilon$. As f has an isolated singular value at 0, f will be a submersion for all these x, so $f^{-1}(f(x))$ will be a manifold. Therefore, each manifold $f^{-1}(f(x))$ will be transversal to the sphere $\mathbb{S}^{n-1}_\varepsilon$. □

This tells us that if a function f satisfies the Thom a_f-condition, then f has the transversality condition from Definition 1.4.5 on page 7. Specifically, if $f : \mathbb{C}^n \to \mathbb{C}$ is a holomorphic function germ with $f(0) = 0$, then f satisfies the transversality condition.

The transversality condition will turn out to be one of the properties needed to show the existence of both the Milnor fibration and the Milnor-Lê fibration. In the litterature of Milnor fibrations, it has therefore been customary to assume that f satisfies the Thom a_f-property, as this is sufficient. However, as we shall see, it is not necessary.

Let us go back to our earlier example 1.4.9 on page 10, and see if this function satisfies the Thom a_f-condition.

Example 2.3.9. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(x, y, z) = (x, y(x^2 + y^2) + xz^2)$. We have that

$$V(f) = \{(x, y, z) \in \mathbb{R}^3 | x = 0 \wedge y = 0\}.$$

We can find a Whitney stratification of $V(f)$ given by $\mathcal{S} = \{S_0, S_+, S_-\}$, where

$$\begin{aligned} S_0 &= \{(0, 0, 0)\}, \\ S_+ &= \{(0, 0, z) | z > 0\}, \\ S_- &= \{(0, 0, z) | z < 0\}. \end{aligned}$$

Then, for a point $p = (0, 0, z) \in S_\pm$, with $z \neq 0$, we have that $T_pS_\alpha = \operatorname{span}\{(0, 0, 1)\}$.

For any point $q \in \mathbb{R}^3 \setminus V(f)$, we have that $T_q f^{-1}(f(q))$ are given by all vectors normal to both ∇P_1 and ∇P_2, where $P_1 = x$ and $P_2 = y(x^2 + y^2) + xz^2$. We compute $\nabla P_1 \times \nabla P_2$.

$$\begin{aligned}\nabla P_1 \times \nabla P_2 &= \begin{bmatrix}1 & 0 & 0\end{bmatrix} \times \begin{bmatrix}2xy + z^2 & x^2 + 3y^2 & 2xz\end{bmatrix} \\ &= \begin{bmatrix}0 & -2xz & x^2 + 3y^2\end{bmatrix}\end{aligned}$$

Now consider the sequence $p_n = (\frac{1}{n}, 0, 1)$ converging to $p = (0, 0, 1)$. Then the tangent space $T_{p_n} f^{-1}(f(p_n))$ is spanned by the vector $u_n = (0, \frac{-2}{n}, \frac{1}{n^2})$. The norm of u_n is $\|u_n\| = \sqrt{\frac{4}{n^2} + \frac{1}{n^4}}$, so if we normalize u_n we get $v_n = (0, \frac{-2}{\sqrt{4 + \frac{1}{n^2}}}, \frac{1}{\sqrt{4n^2+1}})$.

We now see that if we let $n \to \infty$, then $v_n \to (0, 1, 0)$. So the tangent spaces $T_{p_n} f^{-1}(f(p_n))$ converge to the space $T = \mathrm{span}\{(0, 1, 0)\}$, and we have that $T_p S_+ \not\subset T$. Therefore, f does not satisfy the Thom a_f-conditon.

Chapter 3

Milnor Fibrations

3.1 The Milnor-Lê fibration

Much of the information in this chapter is either directly or indirectly from [Sea07].

We want to start by studying the Milnor-Lê fibration for holomorphic function germs. Milnor showed the existence of this fibration for the case where he had an isolated singular point in [Mil68, Chapter 11]. Lê proved that you had a fibration in a more general case in [Lê77].

As we are only interested in how a function behaves close to 0, we will work on function germs throughout this chapter. This implies that while f may not be defined everywhere on $\mathbb{C}^n$, there exists a small open neighborhood around 0 where f is defined. We then just make sure that everything we define is contained inside this neighborhood.

Theorem 3.1.1. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function germ with $f(0) = 0$. Let $\varepsilon > 0$ small enough and $\delta > 0$, $\varepsilon >> \delta$. Let*

$$E(\varepsilon, \delta) = f^{-1}(\overline{\mathbb{B}}_\delta^2 \setminus 0) \cap \overline{\mathbb{B}}_\varepsilon^n$$

and

$$T(\varepsilon, \delta) = f^{-1}(\mathbb{S}_\delta^1) \cap \overline{\mathbb{B}}_\varepsilon^n$$

Then both

$$f|_{E(\varepsilon,\delta)} : E(\varepsilon, \delta) \to \overline{\mathbb{B}}_\delta^2 \setminus 0$$

and

$$f|_{T(\varepsilon,\delta)} : T(\varepsilon, \delta) \to \mathbb{S}_\delta^1$$

are C^∞ fiber bundles.

Proof. As f is a function germ into $\mathbb{C}$, we have from [Hir77] and 2.3.8 on page 20 that f satisfies the transversality condition, and from 2.2.4 on page 17 that $0 \in \mathbb{C}$

is an isolated singular value. We can therefore choose ε and δ so small that f is a proper submersion on both $E(\varepsilon, \delta)$ and $T(\varepsilon, \delta)$, and such that fibers $f^{-1}(t)$ for $0 < \|t\| \leq \delta$ will intersect $\mathbb{S}_\varepsilon^{n-1} \cap f^{-1}(\overline{\mathbb{B}}_\delta^m \setminus 0)$ transversally. Then f will be a proper submersion on both $\partial E(\varepsilon, \delta)$ and $\partial T(\varepsilon, \delta)$ as well. We can now use the Ehresmann's theorem for manifolds with boundary, 1.2.3 on page 3, to conclude that these are indeed C^∞ fiber bundles. $\square$

We will call $T(\varepsilon, \delta)$ for the **Milnor tube**.

We needed only two properties to prove this theorem. The first was that 0 was an isolated singular value and the second was the transversality condition. In the general case we will also need the condition that the dimension of the domain is higher or equal to the dimension of the target, as we need this to get an isolated singular value. We can use this information to formulate an equivalent theorem for real-valued functions.

Theorem 3.1.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $n \geq m \geq 2$, be an analytic function germ with $f(0) = 0$, an isolated singular value at 0, and which satisfies the transversality condition. Let $\varepsilon > 0$ be small enough and $\delta > 0$, $\varepsilon >> \delta$. Let*

$$E(\varepsilon, \delta) = f^{-1}(\overline{\mathbb{B}}_\delta^m \setminus 0) \cap \overline{\mathbb{B}}_\varepsilon^n$$

and

$$T(\varepsilon, \delta) = f^{-1}(\mathbb{S}^{m-1}) \cap \overline{\mathbb{B}}_\varepsilon^n.$$

Then both

$$f|_{E(\varepsilon,\delta)} : E(\varepsilon, \delta) \to \overline{\mathbb{B}}_\delta^m \setminus 0$$

and

$$f|_{T(\varepsilon,\delta)} : T(\varepsilon, \delta) \to \mathbb{S}_\delta^{m-1}$$

are C^∞ fiber bundles.

In other works, one has often either asked for an isolated singular point or for the Thom a_f-condition. As we have seen, an isolated singular point implies the Thom a_f-condition and the Thom a_f-condition implies the transversality condition. These cases are therefore contained in this theorem. In all cases, the Thom a_f-condition has only been needed to obtain the transversality condition. In [AdSCT13] the transversality condition was used to prove the existence of the Milnor fibration (see 3.3.3 on page 28), and we have seen that in general it is sufficient to ask for the transversality condition over the Thom a_f condition.

In Milnor's original book, [Mil68], he asked whether there existed any "non-trivial" examples of such real-valued functions. Examples were hard to find as he wanted 0 to be an isolated singular point. For $m = 2$ and n even, Looijenga proved in [Loo71] that such functions did indeed exist, and it was proved in [CL75] by Church and Lamotke for $m = 2$ and n odd.

3.2 Inflation to the sphere

While fiber bundles on the Milnor-tube are interesting by themselves, fiber bundles are easier to work with if they are defined on more well-known manifolds. Luckily, we can "inflate" the Milnor-tube towards the sphere and by filling some gaps, we get a fibration $\mathbb{S}_\varepsilon^{n-1} \setminus V(f)$. But to do this, we need the following lemma.[Mil68, Lemma 3.4]

Lemma 3.2.1. *If f and g are non-negative analytic function germs from $\mathbb{R}^m \to \mathbb{R}$ which vanish at 0, then there exists an ε such that, for $x \in \overline{\mathbb{B}}_\varepsilon$, the two gradients $\nabla f(x)$ and $\nabla g(x)$ cannot point in exactly opposite directions unless at least one of them vanishes.*

We want a vector field v in $\overline{\mathbb{B}}_\varepsilon \setminus f^{-1}(\mathbb{B}_\delta)$ such that both

$$\langle v(x), x\rangle > 0$$

and

$$\langle v(x), \nabla\|f(x)\|^2\rangle > 0.$$

From the previous lemma we have that $2x = \nabla\|x\|^2$ and $\nabla\|f(x)\|^2$ cannot point in opposite directions, and as they both are nonzero in $\mathbb{B}_\varepsilon \setminus f^{-1}(0)$, we can construct our vector field v by setting

$$v(x) = \frac{x}{\|x\|} + \frac{\nabla\|f(x)\|^2}{\|\nabla\|f(x)\|^2\|}.$$

We can now move along the flow of this vector field from the Milnor-tube to the sphere to get a fiber bundle on $\mathbb{S}_\varepsilon^{n-1} \setminus f^{-1}(\mathbb{B}_\delta^m)$. If we compose this with the map $y \mapsto y/\|y\|$, $y \in \mathbb{S}_\delta^{m-1}$, this is a new fiber bundle. This fiber bundle will then go from $\mathbb{S}_\varepsilon^{n-1} \setminus f^{-1}(\mathbb{B}_\delta^m)$ to $\mathbb{S}^{m-1}$. Let's call this bundle B.

We already know of a fiber bundle on $\mathbb{S}_\varepsilon^{n-1} \cap f^{-1}(\overline{\mathbb{B}}_\delta^m \setminus 0)$, which we get by restricting the fiber bundle we have on $E(\varepsilon, \delta)$. Again we can compose with $y \mapsto y/\|y\|$, now with $y \in \mathbb{B}_\delta^m \setminus \{0\}$ to get a fiber bundle to the sphere $\mathbb{S}^{m-1}$, as $y \mapsto y/\|y\|$ is a trivial fiber bundle. This fiber bundle agrees with the bundle B on $\mathbb{S}_\varepsilon^{n-1} \cap f^{-1}(\mathbb{S}^{m-1})$, so we can then glue these together in a $\mathbb{C}^\infty$ way to get a fiber bundle from the whole of $\mathbb{S}^{n-1} \setminus V(f)$ to $\mathbb{S}^{m-1}$. We get the following theorem.

Theorem 3.2.2. *Let f be as given in Theorem 3.1.2 on the preceding page. Then there exists a C^∞ fiber bundle from $\mathbb{S}_\varepsilon^{n-1} \setminus V(f)$ to $\mathbb{S}^{m-1}$.*

For most of this fiber bundle we have no explicit description of the function, but for $x \in \mathbb{S}_\varepsilon^{n-1} \cap f^{-1}(\overline{\mathbb{B}}_\delta^m \setminus 0)$, it's given by $f(x)/\|f(x)\|$. Under certain conditions, we are able to find a fiber bundle on the whole of $\mathbb{S}^{n-1} \setminus V(f)$ in such a way that the projection is given by $f(x)/\|f(x)\|$ everywhere. Under these conditions, the fiber bundle is called the Milnor fibration.

3.3 The Milnor fibration

The existence of this fiber bundle when $f : \mathbb{C}^n \to \mathbb{C}$ is a holomorphic function was one of the main results in [Mil68]. It was proved by showing that the map $\phi : \mathbb{S}^{2n-1}_\varepsilon \setminus V(f) \to \mathbb{S}^1$, given by $\phi(z) = f(z)/\|f(z)\|$, has no critical points, and that you can construct a vector field transversal to the fibers of ϕ which moves at a constant rate with respect to the argument of $\phi(z)$. This gives the fibers a product structure. We will here use a different proof, instead using the vector fields from the Milnor tube to the sphere.

We start by looking at the more general case, $f : \mathbb{R}^n \to \mathbb{R}^m$, $n \geq m$. Let y be a point on $\mathbb{S}^{m-1}$, and let $X_y = (f/\|f\|)^{-1}(y)$. Then if the vector field from the tube to the sphere is contained in a fitting T_pX_y we see that $f/\|f\|$ will be constant along each integral line. Then the fiber bundle given in Theorem 3.2.2 on the preceding page will be described by $f/\|f\|$ on the whole of $\mathbb{S}^{2n-1}_\varepsilon \setminus V(f)$, just as we want. We therefore need to show when such vector fields exists.

Theorem 3.3.1. *Let f be as given in Theorem 3.1.2 on page 24. We will write $f = (P_1, \ldots, P_m)$. Assume f is d-regular and let $\Lambda = \sum_{j=1}^m P_j \nabla P_j$. Let $\pi(v)$ be the projection from $T_p\mathbb{R}^n$ down to the tangent space $T_pX_{f(p)/\|f(p)\|}$. Then there exists a smooth vector field $v(p)$ on $\overline{\mathbb{B}}^n_\varepsilon \setminus f^{-1}(\mathbb{B}^m_\delta)$ such that*

(i) $\langle p, v(p)\rangle > 0$

(ii) $\langle \Lambda(p), v(p)\rangle > 0$

(iii) $v(p) \in T_pX_{f(p)/\|f(p)\|}$

if and only if

$$\pi(p) = a\pi(\Lambda(p)), a > 0$$

for all $p \in M(f) \setminus f^{-1}(\mathbb{B}^m_\delta)$, where $M(f)$ is the Milnor set of f.

Proof. Assume without loss of generality that p is a point such that $P_m(p) \neq 0$. Let $\Omega_i = P_m \nabla P_i - P_i \nabla P_m$. First, we recognize that $T_pX_{f(p)/\|f(p)\|}$ is the space normal to all Ω_i. If p is not in $M(f)$, then we know that

$$\begin{bmatrix} \Omega_1 \\ \vdots \\ \Omega_{m-1} \\ \Lambda \\ p \end{bmatrix}$$

have rank $m + 1$, so $\pi(p)$ and $\pi(\Lambda(p))$ are linearly independent. We may then find a vector $v(p)$ satisfying the above condtions. This can be done so that it smoothly varies with p by taking $v(p) = \frac{\pi(p)}{\|\pi(p)\|} + \frac{\pi(\Lambda(p))}{\|\pi(\Lambda(p))\|}$.

As f is d-regular, if p is in $M(f)\backslash V(f)$ we must have that $\pi(p)$ and $\pi(\Lambda(p))$ are linearly dependent. Then they either point in the same direction or in opposite directions. If they point in opposite directions, then the set of vectors v in $T_pX_{f(p)}$ with both $\langle p, v\rangle > 0$ and $\langle \Lambda(p), v\rangle > 0$ are empty, so no such vector can exist at p. But if these point in the same direction everywhere on $M(f) \setminus f^{-1}(\mathbb{B}^m_\delta)$, then we can again choose $v(p) = \frac{\pi(p)}{\|\pi(p)\|} + \frac{\pi(\Lambda(p))}{\|\pi(\Lambda(p))\|}$, and $v(p)$ will fulfill the conditions we've set. □

For the case where $f : \mathbb{C}^n \to \mathbb{C}$, $f(0) = 0$, we have that f satisfies everything we need. If we write $f(z) = P(z) + iQ(z)$, we saw in section 1.4 on page 12 that $\Omega_1 = Q\nabla P - P\nabla Q$ and $i\nabla \log f$ are linearly dependent, and we saw that $P\nabla P + Q\nabla Q$ and $\nabla \log f$ point in the same direction. $M(f)$ is where $\nabla \log f(z)$, $i\nabla \log f(z)$ and z are linearly dependent over $\mathbb{R}$, which is where z and $\nabla \log f$ is linearly dependent over $\mathbb{C}$. By Lemma 1.4.11 on page 12, if z and $\nabla \log f$ are linearly dependent over $\mathbb{C}$, then we can write $\nabla \log f = \lambda z$ where $|\arg \lambda| < \frac{\pi}{4}$. If we let $\lambda = a + ib$, we can compute

$$\begin{aligned}(a+ib)z &= \nabla \log f\\ z &= \frac{1}{a+ib}\nabla \log f\\ z &= \frac{a-ib}{a^2+b^2}\nabla \log f\\ z &= \frac{a}{a^2+b^2}\nabla \log f - \frac{b}{a^2+b^2}i\nabla \log f\\ z &= \frac{\cos(\arg\lambda)}{\sqrt{a^2+b^2}}\nabla \log f - \frac{\sin(\arg\lambda)}{\sqrt{a^2+b^2}}i\nabla \log f\end{aligned}$$

As $\cos(\arg \lambda) > 0$, we have that $\pi(\nabla \log f)$ and $\pi(z)$ must point in the same direction, and we get Milnor's fibration theorem:

Theorem 3.3.2. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function germ with $f(0) = 0$. Then, there exist an ε_0 such that for all $\varepsilon_0 > \varepsilon > 0$, there exists a fibration from $\mathbb{S}^{2n-1}_\varepsilon \setminus V(f)$ to $\mathbb{S}^1$ given by $f/\|f\|$.*

Proof. By Theorem 3.3.1 on the facing page we have a vector field v pointing out from the sphere, out from the Milnor-tube and with $f(p)/\|f\|(p)$ constant along the flow lines of the vector field. We then construct a fiber bundle as in 3.2.2 on page 25. This will be given by $f(p)/\|f(p)\|$ everywhere. □

Milnor showed an example in [Mil68, Chapter 11] where we have the Milnor-Lê fibration, but lacked the Milnor fibration. We will look at this function in Example 3.4.7 on page 36. There may exist situations where the opposite is the case, where the Milnor fibration will exist but the Milnor-Lê do not. However, we have no examples of this. The following theorem, from [AdSCT13], gives us a situation where this might happen. Note that we will not assume that f has an isolated singular value at 0, so we can not be certain that the Milnor-Lê fibration exists. It was originally proved by finding a fiber bundle on $\mathbb{S}_\varepsilon^{n-1} \cap f^{-1}(\overline{\mathbb{B}}_\delta^m)$, another fiber bundle on $\mathbb{S}_\varepsilon^{n-1} \setminus f^{-1}(\mathbb{B}_\delta^m)$ and then glue these together along the edge. We will give an alternative proof, along the lines of the proof of the Ehresmann fibration theorem. We can not use the Ehresmannn fibration theorem itself, as $\mathbb{S}_\varepsilon^{n-1} \setminus V(f)$ is not compact, so $f/\|f\||_{\mathbb{S}_\varepsilon^{n-1}\setminus V(f)}$ is not proper.

Theorem 3.3.3. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $n \geq m \geq 2$, be an analytic function germ with $f(0) = 0$, given by $f = (P_1, \ldots, P_m)$. Assume that f satisfies the transversality condition and is d-regular. Let $\varepsilon > 0$ be sufficiently small. Then $\phi : \mathbb{S}_\varepsilon^{n-1} \to \mathbb{S}^{m-1}$ given by $\phi = f/\|f\|$ is a fiber bundle.*

Proof. Let $r = (r_1, \ldots, r_m)$ be a point in $\mathbb{S}^{m-1}$. We can assume without loss of generality that $|r_m| \geq |r_i|$ for all $i = 1, \ldots, m$. Let W be the half-sphere containing r bounded by the hyperplane $r_m = 0$. We may then use the chart $y : W \to \mathbb{R}^{m-1}$ we found in section 1.4 on page 8 given by $y(r_1, \ldots r_m) = (\frac{r_1}{r_m}, \ldots, \frac{r_{m-1}}{r_m})$. We then have from d-regularity and the matrix computations in lemma 1.4.7 on page 9 that the matrix

$$\begin{bmatrix} \Omega_1 \\ \vdots \\ \Omega_{m-1} \\ p \end{bmatrix},$$

where $\Omega_i = P_m \nabla P_i - P_i \nabla P_m$, have rank m for all $p \in \phi^{-1}(W)$.

Let $W' = \phi^{-1}(W)$ and let $p \in W'$. Let $\pi(v)$ be the projection from $T_p\mathbb{R}^n$ to $T_p\mathbb{S}_{\|p\|}^{n-1}$ and define $\Omega_i' = \pi(\Omega_i)$. We see that

$$d(y \circ \phi) \circ \pi = P_m^{-2} \begin{bmatrix} \Omega_1' \\ \vdots \\ \Omega_{m-1}' \end{bmatrix}$$

We want to lift the basis vector fields e_k of $\mathbb{R}^{m-1}$ via $y \circ \phi$ to vector fields ω_k on W'. We can then move along the flow of these lifted vector fields to get a product structure on the fibers, giving us a fiber bundle(see proof of [Dun13, Lemma 9.5.8]).

To make sure we can do this we must show, for a given k and for all points p in the fiber of r, that these flows can be defined on a common interval. We have everything we need if we can define these integral curves on all of $\mathbb{R}$, and we see that the only thing obstructing this is the possibility that a curve $\gamma(t)$ in $\mathbb{S}_\varepsilon^{n-1}$ may reach $V(f)$ in finite time. This happens if and only if $\|f(\gamma(t))\|$ approaches 0 in finite time.

If $\|f(\gamma(t))\|$ approaches 0 in finite time, then $g(t) = \log(\|f(\gamma(t))\|^2)$ must approach $-\infty$ in finite time. This can not happen if $\frac{\mathrm{d}g}{\mathrm{d}t}$ is bounded. We see that

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{2\langle \Lambda(\gamma(t)), \omega_k(\gamma(t))\rangle}{\|f(\gamma(t))\|^2}$$

where $\Lambda(p) = \sum_{j=1}^m P_j(p)\nabla P_j(p)$.

So if we can prove that we can define ω_k in such a way that these are liftings of e_k and a common upper bound on

$$\left| \frac{\langle \Lambda(p), \omega_k(p)\rangle}{\|f(p)\|^2} \right|$$

exists independent of p, then ϕ must be a fiber bundle.

Let $T \subset W$ be all points q with $|q_m| \geq \frac{1}{\sqrt{m}}$. As we assumed that $|r_m| \geq |r_i|$ for all $i = 1, \ldots, m$, we have that $r \in T$. Let $T' = (f/\|f\|)^{-1}(T)$. As $f/\|f\|$ is continuous, T' is closed in $\mathbb{B}^n_\varepsilon \setminus V(f)$.

If p is a point in $\mathbb{S}^{n-1}_\varepsilon \cap T'$ where

$$A = \begin{bmatrix} \Omega'_1(p) \\ \vdots \\ \Omega'_{m-1}(p) \\ \Lambda'(p) \end{bmatrix}$$

have rank m, we can define $\omega_k \in T_p\mathbb{S}_\varepsilon$ such that

$$\begin{aligned} \langle \Omega'_k(p), \omega_k(p)\rangle &= P_m^2(p), \\ \langle \Omega'_i(p), \omega_k(p)\rangle &= 0 \qquad \text{for } i \neq k, \\ \langle \Lambda'(p), \omega_k(p)\rangle &= 0. \end{aligned}$$

The first two equations make sure that ω_k is a lifting of e_k, and the last equation makes sure that $\frac{\mathrm{d}g}{\mathrm{d}t} = 0$. If we insist that ω_k is in $\mathrm{span}\{\Omega'_1, \ldots, \Omega'_{m-1}, \Lambda'\}$, then ω_k is uniquely defined and varies smoothly with p.

Let L be the set of points in $\mathbb{S}^{n-1}_\varepsilon \cap T'$ where the rank of A is less than m. For a p in L, we may find a $\omega_k \in \mathrm{span}\{\Omega'_1, \ldots, \Omega'_{m-1}\}$ such that

$$\begin{aligned} \langle \Omega'_k(p), \omega_k(p)\rangle &= P_m^2(p), \\ \langle \Omega'_i(p), \omega_k(p)\rangle &= 0 \qquad \text{for } i \neq k. \end{aligned}$$

Then ω_k is a lifting of e_k. If we assume that we have a bound M on $\frac{\mathrm{d}g}{\mathrm{d}t}$ for all $p \in L$, then by continuity we may find a neighborhood W_p of p such that if $q \in W_p$ and $\omega_k(q)$ is defined the same way, then

$$\left| \frac{\langle \Lambda(q), \omega_k(q)\rangle}{\|f(q)\|^2} \right| \leq 2M.$$

Then we may find local lifts $\omega_{k,p}$ of e_k such that each is bounded by $2M$. We can patch these together by partition of unity to get a vector field ω_k defined on $\mathbb{S}_\varepsilon^{n-1} \cap T'$. It will be a lift of e_k through $y \circ \phi$ such that integral curves cannot approach $V(f)$ in finite time.

The sphere $\mathbb{S}^{m-1}$ can be covered by $2m$ sets which are defined the way we defined T, two for each possible value of j such that $|r_j| \geq |r_i|$ for all $i = 1, \ldots, m$. So if we have a bound on each of these, we can choose the largest of those to get a bound for all values $p \in \mathbb{B}_\varepsilon^n \setminus V(f)$.

All we need to do now is to prove that for each T there is an upper bound M on L. The vectors $\Omega_i'(p)$ for $i = 1, \ldots, m-1$ and $\Lambda'(p)$ are well defined in all of W', so let N be all points in W' where the matrix A has rank less than m. We now see that the set where the matrix A has rank less than m is the same set as where the matrix

$$\begin{bmatrix} \Omega_1(p) \\ \vdots \\ \Omega_{m-1}(p) \\ \Lambda(p) \\ p \end{bmatrix}$$

have rank less than $m+1$. We recognize this as the set $M(f) \cap W'$. We therefore have

$$\begin{aligned} L &= \mathbb{S}_\varepsilon^{n-1} \cap N \cap T' \\ &= \mathbb{S}_\varepsilon^{n-1} \cap M(f) \cap T' \\ &= \mathbb{S}_\varepsilon^{n-1} \cap (M(f) \setminus V(f)) \cap T'. \end{aligned}$$

As T' is closed in $\mathbb{B}_\varepsilon^n \setminus V(f)$, we have $(\overline{T'} \setminus T') \subset V(f)$. Therefore

$$(M(f) \setminus V(f)) \cap T' = (M(f) \setminus V(f)) \cap \overline{T'}.$$

As $M(f)$ is closed and we have the transversality condition

$$\overline{M(f) \setminus V(f)} \cap V(f) = \{0\},$$

we have that

$$\overline{M(f) \setminus V(f)} = (M(f) \setminus V(f)) \cup \{0\}.$$

If we combine this, we get that

$$\mathbb{S}_\varepsilon^{n-1} \cap (M(f) \setminus V(f)) \cap T' = \mathbb{S}_\varepsilon^{n-1} \cap \overline{M(f) \setminus V(f)} \cap \overline{T'}.$$

Therefore, we can describe L as

$$L = \mathbb{S}_\varepsilon^{n-1} \cap \overline{M(f) \setminus V(f)} \cap \overline{T'}.$$

As all of these are closed, L is compact. Then $\frac{\mathrm{d}g}{\mathrm{d}t}$ is a continuous function on a compact, and therefore bounded. We have the upper bound we needed, so we can patch these vector fields together by partitions of unity to get vector fields $\omega_{k,W'}$ on W', and then by another partition on unity to vector fields ω_k on $\mathbb{S}_\varepsilon^{n-1} \setminus V(f)$. $\square$

3.4 Equivalence of fibrations

Depending on the properties of our analytic function f, we may be in a situation where we can prove the existence of neither the Milnor fibration or the Milnor-Lê fibration, only the Milnor fibration, only the Milnor-Lê fibration or both fibrations. But even when we have both the Milnor-Lê fibration and the Milnor fibration, we can not be certain these fibrations are equivalent.

What we mean by equivalence of these fibrations will have to be adressed. We have that the Milnor-Lê fibration is defined on a compact set, but the Milnor fibration is not. By equivalence of these fibrations we will therefore mean that they are equivalent when the Milnor fibration is restricted to $S_\varepsilon^{n-1} \cap f^{-1}(\mathbb{B}_\delta^m)$. Fortunately, this will uniquely determine the fibration, see proof of [CMSS09, Thm. 2].

As a reminder, earlier we defined the analytic set X_y for all $y \in \mathbb{S}^{m-1}$ as $X_y = (f/\|f\|)^{-1}(y)$, and we defined $\pi(v)$ as the projection from $T_p\mathbb{R}^n$ to $T_pX_{f(p)/\|f(p)\|}$.

When we have both the fibration on the tube and the Milnor fibration on the sphere, we want to know if they are equivalent fibrations. The computations from the last sections paved the way, as we see that what we need is for $\pi(\nabla\|p\|^2) = \pi(2p)$ and $\pi(\nabla\|f(p)\|^2) = \pi(2\sum_{i=1}^m P_i(p)\nabla P_i(p)) = \pi(2\Lambda(p))$ to point in the same direction along all of $M(f) \setminus f^{-1}(\mathbb{B}_\delta^m)$.

If these point in the same direction, we can construct a vector field v such that v points out from the sphere, out from the tube and such that $f/\|f\|$ is constant along the flow of the vector field. We may then construct an equivalence of fibrations the same way we did in Theorem 3.3.2 on page 27.

Let K be a connected component of $M(f) \setminus V(f)$. If $\pi(p)$ and $\pi(\Lambda(p))$ point in the same direction at a point p in K then by continuity they must point in the same direction for all q in K. So we need to show when we can expect the vectors $\pi(p)$ and $\pi(\Lambda(p))$ to point in the same direction for at least one point in each connected component of $M(f) \setminus V(f)$. We have the following lemma:

Lemma 3.4.1. *If there exists a $\nu > 0$, an $y \in \mathbb{S}^{m-1}$ and an analytic curve $\gamma : [0, \nu) \to \mathbb{R}^n$ with $\gamma(0) = 0$ and $\gamma((0, \nu)) \subset K \cap X_y$, then $\pi(p)$ and $\pi(\Lambda)$ point in the same direction for all $p \in K$.*

Proof. As γ is analytic, there exists an $0 < \varepsilon_1 < \nu$ such that $\frac{\mathrm{d}\|\gamma(t)\|^2}{\mathrm{d}t} > 0$ for all $t < \epsilon_1$. Since f is analytic, $f(\gamma(t))$ is also an analytic curve, so there exists an $0 < \varepsilon_2 < \nu$ such that $\frac{\mathrm{d}\|f(\gamma(t))\|^2}{\mathrm{d}t} > 0$ for all $t < \varepsilon_2$.

Choose any t with $0 < t < \min(\varepsilon_1, \varepsilon_2)$. We then have

$$\frac{\mathrm{d}\|\gamma(t)\|^2}{\mathrm{d}t} = 2\langle \gamma'(t), \gamma(t)\rangle > 0$$

and

$$\frac{\mathrm{d}\|f(\gamma(t))\|^2}{\mathrm{d}t} = 2\langle \gamma'(t), \Lambda(\gamma(t))\rangle > 0.$$

As $\gamma((0,\nu)) \subset X_y$, we must have $\gamma'(t) \in T_{\gamma(t)}X_y$, so we can project $\gamma(t)$ and $\Lambda(\gamma(t))$ by π to $T_{\gamma(t)}X_y$ without changing the inner products. For $p = \gamma(t)$, we can now se that we have a vector $\gamma'(t)$ with $\langle \gamma'(t), \pi(p)\rangle > 0$ and $\langle \gamma'(t), \pi(\Lambda(p))\rangle > 0$. This is impossible if $\pi(p)$ and $\pi(\Lambda(p))$ point in opposite directions. Therefore $\pi(p)$ must point in the same direction as $\pi(\Lambda(p))$ at p and by continuity this must be true for all $p \in K$. □

We then have equivalence of the two fibrations if, for any component K of $M(f) \setminus V(f)$ with $K \cap \mathbb{B}^n_\varepsilon \neq \emptyset$, such an analytic curve exists.

From Theorem 2.2.2 on page 16, as $M(f) \setminus V(f)$ is a semianalytic set, we have that the connected components are also semianalytic and the family of connected components are locally finite. We may therefore assume that K contains points arbitrarily close to the origin, as otherwise we could shrink ε until $K \cap \mathbb{B}^n_\varepsilon = \emptyset$. After shrinking ε finitely many times, we have that the remaining components will contain points arbitrarily close to the origin (possibly vacuously).

If f behaves nicely on each K, we can show that such a curve must exist.

Lemma 3.4.2. *Assume f satisfies the transversality condition. If f is an open map when restricted to K, then such an analytic curve exists.*

Proof. As the only point in the intersection of $V(f)$ and $\overline{K}$ is 0, we can choose a $\varepsilon_0 > 0$ in such a way that $\varepsilon_0 < \|y\|$ for all $y \in f(K \cap \mathbb{S}^{n-1}_\varepsilon)$. Choose a sequence in $K \cap \mathbb{B}^n_\varepsilon$ converging to 0. The image of this sequence with respect to f must converge to 0 in the target.

Therefore there must exist a point $y_0 = f(x_0)$ with $x_0 \in K$ and $y_0 \in \mathbb{B}^m_{\varepsilon_0}$. Let ℓ_{y_0} be the line through y_0 and the origin, and let $\widetilde{\ell}_{y_0}$ be the interval $\{ty_0 | 0 < t < \frac{\varepsilon_0}{\|y_0\|}\}$. Let $N = \widetilde{\ell}_{y_0} \cap f(K \cap \mathbb{B}^n_\varepsilon)$. We want to show that $N = \widetilde{\ell}_{y_0}$.

Choose $y \in N$ and an open neighborhood U of y. Then $f^{-1}(U) \cap K \cap \mathbb{B}^n_\varepsilon$ is open in K, as f is continuous and $\mathbb{B}^n_\varepsilon$ is open. As f is open when restricted to K, $f(f^{-1}(U) \cap K \cap \mathbb{B}^n_\varepsilon)$ is open and contained in $f(K \cap \mathbb{B}^n_\varepsilon)$. The intersection of this with $\widetilde{\ell}_{y_0}$ is open in $\widetilde{\ell}_{y_0}$ and contained in N. So N is an open subset of $\widetilde{\ell}_{y_0}$.

Assume now $y \in \overline{N} \setminus N$, where the closure is taken inside of $\widetilde{\ell}_{y_0}$. Then there exists a sequence $y_n \in N$ converging to $y \in \widetilde{\ell}_{y_0}$, and we can find $x_n \in K \cap \mathbb{B}^n_\varepsilon$ with $f(x_n) = y_n$. As $\overline{K \cap \mathbb{B}^n_\varepsilon}$ is compact, we can find a convergent subsequence of x_n, converging to an $x \in \overline{K \cap \mathbb{B}^n_\varepsilon}$. By continuity, we have that $f(x) = y$. But if $x \in K \cap B^n_\varepsilon$, then we have $y \in N$. As we assumed $y \notin N$, then we have $x \notin K \cap \mathbb{B}^n_\varepsilon$. So either we have $x \in \mathbb{S}^{n-1}_\varepsilon$ or we have $x = 0$. Both of these options are impossible, as $f(0) = 0 \notin \widetilde{\ell}_{y_0}$ and $f(\mathbb{S}^{n-1}_\varepsilon) \cap \widetilde{\ell}_{y_0} = \emptyset$. We conclude that $\overline{N} \setminus N = \emptyset$, which means that N is closed as a subset of $\widetilde{\ell}_{y_0}$.

Now N is a nonempty open and closed subset of $\widetilde{\ell}_{y_0}$, which is connected, so we have $N = \widetilde{\ell}_{y_0}$. Choose any sequence in $\widetilde{\ell}_{y_0}$ converging to 0. This will give us a sequence in $K \cap \mathbb{B}^n_\varepsilon \cap f^{-1}(\ell_{y_0})$. We can again find a convergent subsequence in the closure, which must converge to a point in $V(f)$. As $\overline{K} \cap V = \{0\}$, it must converge to 0. If we let $y_1 = \frac{y_0}{\|y_0\|}$, we can see that $f^{-1}(\ell_{y_0}) \subset X_{y_1}$. Therefore $K \cap X_{y_1}$ is a semianalytic set with 0 in the closure. We can then use the Curve Selection Lemma (Lemma 2.2.3 on page 17) to find an analytic curve satisfying our requirements. $\square$

Luckily for us, there is a large family of singularities which nets us exactly what we need.

Lemma 3.4.3. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be an analytic function germ satisfying d-regularity and the transversality condition with isolated singular value at* 0. *Let $F(p) = (f(p), \|p\|^2)$, and assume that F is a submersion with folds on $\mathbb{B}^n_\varepsilon \setminus V(f)$. Then $\pi(p)$ and $\pi(\Lambda(p))$ point in the same direction for all $p \in M(f) \setminus V(f)$.*

Proof. We have from Lemma 1.5.4 on page 13 that in this situation f is a local diffeomorphism on $M(f) \setminus V(f)$, and therefore also open. $\square$

Let us have a last look at the example which have been following us throughout the thesis, last seen as example 2.3.9 on page 20. We will see that this will give us equivalent fibrations.

Example 3.4.4. Let $f(x, y, z) = (x, y(x^2 + y^2) + xz^2)$. We saw in example 1.3.4 on page 5 that f had an isolated singular value. In example 1.4.9 on page 10 we saw that f satisfied the transversality condition and was d-regular. This tells us that both the Milnor fibration and the Milnor-Lê fibration exists. All we need to show is that these are equivalent. We will show this by proving that f is a local diffeomorphism when restricted to $M(f)\setminus V(f)$. For if f is a local diffeomorphism, then f is open and by lemma 3.4.2 on the facing page and lemma 3.4.1 on page 31 we will have an equivalence of the fibrations.

We have earlier computed that $M(f) \setminus V(f) = \{z = 0 \wedge (x, y) \neq (0, 0)\}$. As $z = 0$ everywhere on $M(f) \setminus V(f)$, $f|_{M(f)\setminus V(f)}$ is given by $f(x, y) = (x, y(x^2 + y^2)$. We get

$$df|_{M(f)\setminus V(f)} = \begin{bmatrix} 1 & 0 \\ 2xy & x^2 + 3y^2 \end{bmatrix}.$$

This matrix has rank 2 as long as $x^2 + 3y^2 \neq 0$, i.e. $(x, y) \neq (0, 0)$. So f is a local diffeomorphism on $M(f) \setminus V(f)$.

We end the thesis by showing some extra examples with computations.

Example 3.4.5. Let $f(x_1, x_2, x_3, x_4) = (2x_1x_3 + 2x_2x_4, 2x_2x_3 - 2x_1x_4, x_1^2 + x_2^2 - x_3^2 - x_4^2)$. If (x_1, x_2, x_3, x_4) is a zero of this function, we must have

$$\begin{aligned} x_1^2 + x_2^2 &= x_3^2 + x_4^2 \\ 2x_2x_3 &= 2x_1x_4 \\ 2x_1x_3 &= -2x_2x_4. \end{aligned}$$

If two of these variables are zero, all of them must be zero. For if either both $x_1 = x_2 = 0$ or $x_3 = x_4 = 0$, we have from the first equation that the remaining two must be zero. And if x_i and x_j are zero, where $i = 1, 2$ and $j = 3, 4$, the first equation gives us $x_k^2 = x_l^2$ and one of the two others give us $x_k x_l = 0$, where k and l are the two remaining indices. These imply that either x_k or x_l is 0, and therefore both must be 0, again from the first equation.

We can multiply together the bottom two equations either by multiplying the left hand side with the left hand side and the right hand side with the right hand side, or by multiplying the left hand side of the first with the right hand side of the second and the right hand side of the first by the left hand side of the second. This gives us two equations.

$$\begin{aligned} 4x_1x_2(x_3^2 + x_4^2) = 0 \\ 4x_3x_4(x_1^2 + x_2^2) = 0 \end{aligned}$$

We see that for both of these to be zero, we need two of the variables $x_i, i = 1, 2, 3, 4$ to be zero, which means that all of them must be. The zero set of f is therefore only $(0, 0, 0, 0)$.

We compute the jacobian of f and get

$$df = \begin{bmatrix} 2x_3 & 2x_4 & 2x_1 & 2x_2 \\ -2x_4 & 2x_3 & 2x_2 & -2x_1 \\ 2x_1 & 2x_2 & -2x_3 & -2x_4 \end{bmatrix}.$$

The function f is singular in a point if all four minors of this matrix has rank less than three. We compute when the determinants of these minors are zero, and get four equations.

$$\begin{aligned} 8x_1(x_1^2 + x_2^2 + x_3^2 + x_4^2) = 0 \\ 8x_2(x_1^2 + x_2^2 + x_3^2 + x_4^2) = 0 \\ 8x_3(x_1^2 + x_2^2 + x_3^2 + x_4^2) = 0 \\ 8x_4(x_1^2 + x_2^2 + x_3^2 + x_4^2) = 0 \end{aligned}$$

For all of these to be zero, we see that we must have $x_1 = x_2 = x_3 = x_4 = 0$. So the set of singular points of f is again only the point $(0, 0, 0, 0)$. This is an isolated critical point, so the set satisfies the transversality condition.

If p is a nonzero point in $\mathbb{R}^4$ and a is a real number, we see that $f(ap) = a^2 f(p)$. Therefore, if ℓ is the line through both p and the origin, we must have that ℓ is contained in $(f/\|f\|)^{-1}(f(p)/\|f(p)\|)$, as $\frac{f(ap)}{\|f(ap)\|} = \frac{a^2 f(p)}{a^2\|f(p)\|} = f(p)/\|f(p)\|$. Therefore, the Milnor set $M(f/\|f\|)$ must be empty, so f is d-regular.

As we have isolated singular value, the transversality condition and d-regularity, we must have both the Milnor-Lê fibration and the Milnor-fibration. The Milnor set $M(f)$ is where the matrix

$$\begin{bmatrix} 2x_3 & 2x_4 & 2x_1 & 2x_2 \\ -2x_4 & 2x_3 & 2x_2 & -2x_1 \\ 2x_1 & 2x_2 & -2x_3 & -2x_4 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix}$$

has less than rank 4. We can reduce this to

$$\begin{bmatrix} 0 & 0 & x_3 & x_4 \\ x_1 & x_2 & 0 & 0 \\ x_3 & x_4 & x_1 & x_2 \\ x_4 & -x_3 & -x_2 & x_1 \end{bmatrix}.$$

We compute the determinant and get

$$x_3(x_1(x_1x_4+x_2x_3)-x_2(x_1x_3-x_2x_4))-x_4(x_1(-x_2x_4+x_1x_3)-x_2(-x_2x_3-x_1x_4) = 0.$$

Therefore, $M(f) = \mathbb{R}^4$. $M(f)\backslash V(f)$ is therefore a single component, and if ℓ is the line through both p and the origin, the connected component of $\ell\setminus\{0\}$ containing p will be an analytic curve contained in the set $X_{f(p)/\|f(p)\|} \cap (M(f) \setminus V(f))$, as previously noted. Therefore, the Milnor fibration and the Milnor-Lê fibration will be equivalent.

This example is in fact the well-known Hopf-fibration from $\mathbb{S}^3$ to $\mathbb{S}^2$. We could instead have computed that $\|f(p)\| = \|p\|^2$, so $f|_{\mathbb{S}^3_\varepsilon} : \mathbb{S}^3_\varepsilon \to \mathbb{S}^2_{\varepsilon^2}$. We can see that this is a proper submersion, so by Ehresmann's fibration theorem this is indeed a fibration. That the Milnor fibration and the Milnor-Lê fibration is equivalent is then of no surprise, as they only differ by multiplication by a constant.

Example 3.4.6. Let $f(x, y, z) = (x^2 + y^2, (x^2 + y^2)z)$. For this to be zero, we must have $x^2 + y^2 = 0$, so $V(f) = \{(x, y, z) \in \mathbb{R}^3 | x = 0 \wedge y = 0\}$. We compute

$$df = \begin{bmatrix} 2x & 2y & 0 \\ 2xz & 2yz & x^2 + y^2 \end{bmatrix}.$$

If (x, y, z) is a singular point, then all 2×2 minors of this matrix must have determinant equal to 0. This gives us the equations,

$$2x(x^2 + y^2) = 0$$
$$2y(x^2 + y^2) = 0$$

which are zero when $x = 0$ and $y = 0$. We have $\Sigma f = V(f)$. If we compute

$$\begin{aligned} \Omega_1 &= z(x^2 + y^2) \begin{bmatrix} 2x & 2y & 0 \end{bmatrix} - (x^2 + y^2) \begin{bmatrix} 2xz & 2yz & x^2 + y^2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & (x^2 + y^2)^2 \end{bmatrix} \end{aligned}$$

we can use this to see if f is d-regular. We see that the matrix

$$\begin{bmatrix} 0 & 0 & (x^2+y^2)^2 \\ x & y & z \end{bmatrix}$$

has a rank less than 2 only when $x = 0$ and $y = 0$. Therefore f is d-regular as well. However, if we compute the determinant of

$$\begin{bmatrix} 2x & 2y & 0 \\ 2xz & 2yz & x^2+y^2 \\ x & y & z \end{bmatrix}$$

we get 0, so we have $M(f) = \mathbb{R}^3$. Therefore, $\overline{M(f) \setminus V(f)} \cap V(f) = V(f)$ and f does not satisfy the transversality condition. We can therefore not prove that f is a fibration. Indeed, we see that the inverse image $f^{-1}(r,s)$ of a point (r,s) on $\mathbb{S}^1$ is the set of points where $x^2+y^2 = r$ and $rz = s$. This is a circle with a constant $z = r/s$ if $r > 0$ and the empty set if $r \leq 0$. Therefore, we will have that if (r,s) is a point on $\mathbb{S}^1 \subset \mathbb{R}^2$ with $r \neq 0$, $(f/\|f\|)^{-1}(r,s)$ can not be diffeomorphic to $(f/\|f\|)^{-1}(-r,s)$, as one of these must be the empty set, while the other is nonempty.

Example 3.4.7. Let $f(x,y) = (x, x^2 + y(x^2+y^2))$. We see that for $f(x,y)$ to be 0, we must have $x = 0$ and $y = 0$. We compute the matrix

$$df = \begin{bmatrix} 1 & 0 \\ 2x+2xy & x^2+3y^2 \end{bmatrix}.$$

For this to have rank less than 2, we must have $x^2+3y^2 = 0$. Therefore, we have $\Sigma f = V(f)$. This is an isolated singular point, so our function f satisfies the transversality condition. We compute Ω_1 to be

$$\begin{aligned} \Omega_1 &= (x^2 + y(x^2+y^2) \begin{bmatrix} 1 & 0 \end{bmatrix} - x \begin{bmatrix} 2x+2xy & x^2+3y^2 \end{bmatrix} \\ &= \begin{bmatrix} y^3 - x^2 - x^2y & -x^3 - 3xy^2 \end{bmatrix} \end{aligned}$$

and use this to check whether f is d-regular. To check this, we must compute the determinant of

$$\begin{bmatrix} y^3 - x^2 - x^2y & -x^3 - 3xy^2 \\ x & y \end{bmatrix}$$

which is

$$x^4 + 2x^2y^2 + y^4 - x^2y.$$

We check where the determinant is zero by substituting z for x^2. This gives us the equation

$$z^2 + (2y^2 - y)z + y^4 = 0$$

This equation has real solutions if $(2y^2-y)^2-4y^4 = y^2(1-4y) \geq 0$, i.e. $y \leq \frac{1}{4}$. If $y \geq 0$, the solutions are non-negative, so we can solve for x when $y \in [0, \frac{1}{4}]$. As we see that $x = 0 \Leftrightarrow y = 0$, this gives us nonzero solutions of (x, y) arbitrarily close to the origin. Our function f is therefore not d-regular.

As we have isolated singular value and the transversality condition, we can conclude that f works as the Milnor-Lê fibration. However, as f is not d-regular, we have not proved the existence of the Milnor-fibration.

This last example was mentioned by Milnor in [Mil68, Ch. 11] as an example of a function which had the Milnor-Lê fibration, but where we could not get the Milnor fibration.

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